QT209: Introduction to Quantum Communication and Cryptography

(August Semester 2025-26)

Instructors: Manukumara Manjappa (IAP)

Sanjit Chatterjee (CSA)

Varun Raghunathan (ECE)

Class Timing, venue and intended students

- Timing: Monday, Wednesday 10:00-11:30 AM
- ECE department, Room MP30

- Students: Core course for M.Tech. - QT programme
  Elective for M.Tech. and Ph.D. students
  Undergraduate students

Syllabus

Prof. Manukumara Manjappa:

1. Optics Introduction
2. Wave motion (SHM), Phase and Group velocity, wave propagation and wave equation, Huygens theory
3. Electromagnetic nature of light: Polarization, double refraction (QWPs and HWPs), Maxwell's equations, Poynting vectors, The Continuity conditions, Plane waves in dielectric medium, Total internal reflection and evanescent waves
4. Interference phenomena: two slit interference, coherence, Interferometers (Michelson and F-P)
5. Lasers, Photodetectors and Fiber optics: Laser basics, photodetectors, Fiber basics numerical aperture, Attenuation in optical fibers, multimode and single mode fibers (basics).

Prof. Sanjit Chatterjee:

6. Cryptography and one-time pad
7. Public and private key cryptography
8. Quantum key distribution
9. Quantum cryptography.

Prof. Varun Raghunathan:

10. Quantum versions of classical devices - single photon sources, entangled photon sources, number states, coherent states, fiber and free-space channel, single photon detectors
11. Implementation of BB84 and E91 protocols, implementational non-idealities, side-channels and possible mitigations strategies

Pre-requisites

- Basic linear algebra, differential equations, statistics and probability
- Basic concepts of Number Theory and Algorithm

- (Please fill more details)

Reference books

- Gerry and Knight, Introduction to Quantum Optics
- Mark Fox, Quantum Optics
- Neilsen and Chuang, Quantum Computation and Quantum Information
- Ajoy Ghatak, Optics, 6 ed. McGraw Hill Education (India) Pvt Ltd. 2016.
- Eugene Hecht, Optics, 4^{th} ed. Pearson Education, Inc., 2002.
- David J Griffiths, Introduction to electrodynamics, Prentice Hall 1999
- Katz and Lindell, Introduction to modern cryptography, 2nd ed.

- (Please fill more details)

Exams and evaluation

- 2 in-class tests/ quizzes
- 1 take-home assignment. - 60% weightage
- 1 final exam (in class) – 40% weightage

Academic Integrity

- Ethical integrity is essential in all human endeavours of excellence….A flourishing academic environment entails rigorous and sincere adherence to ethical practices. Therefore, it is important that the researchers and students in the Institute are sensitized in this matter, and are informed about the commonly recognized unacceptable behaviours in classes, research and research-communications.
- Refer to "**IISc Policy for Academic Integrity in Research"** for more details.

- Cheating: is unacceptable academic behaviour and may be classified into different categories:
  - Copying during exams, and copying of homework assignments, term papers or manuscripts. Allowing or facilitating copying, or writing a report or exam for someone else.
  - Using unauthorized material, copying, collaborating when not authorized, and purchasing or borrowing papers or material from various sources.
  - Fabricating (making up) or falsifying (manipulating) data and reporting them in thesis and publications.

- **Cheating in class tests, take-home exams, final exam will be taken very seriously and will be severely penalized**

QT-209: Introduction to Quantum Communications and Cryptography
(Aug-Dec 2025),
Mondays and Wednesdays, 10:00-11:30AM

Classical Optics

→ 9 class

Instructor: Dr. Manukumara Manjappa

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COURSE CONTENTS

- **Optics Introduction**
- **Wave motion (SHM)**, Phase and Group velocity, Dispersion, wave propagation and wave equation, Huygens theory
- **Electromagnetic nature of light**: Polarization, double refraction (QWPs and HWPs), Maxwell's equations, Poynting vectors, The Continuity conditions, Plane waves in dielectric medium, Total internal reflection and evanescent waves
- **Interference phenomena**: Coherence, Interferometers (Michelson and F-P)

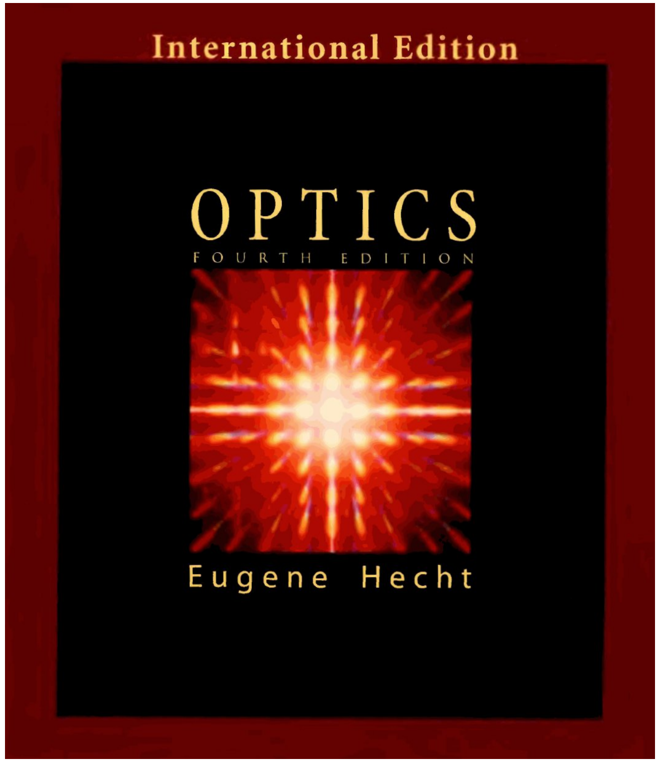
SL, FL silicon

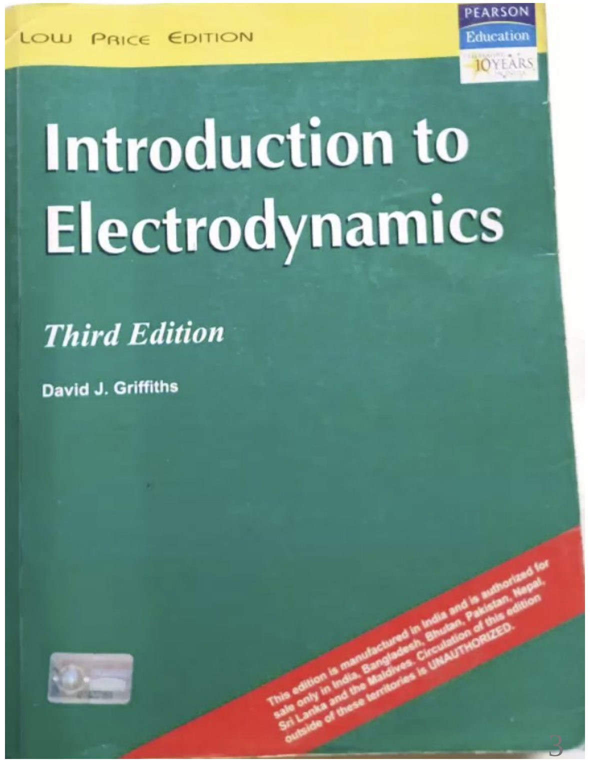
- **Lasers, Photodetectors and Fiber optics**: Laser basics, photodetectors, Fiber basics numerical aperture, Attenuation in optical fibers, multimode and single mode fibers (basics).

dispersion

BOOKS/REFERENCES

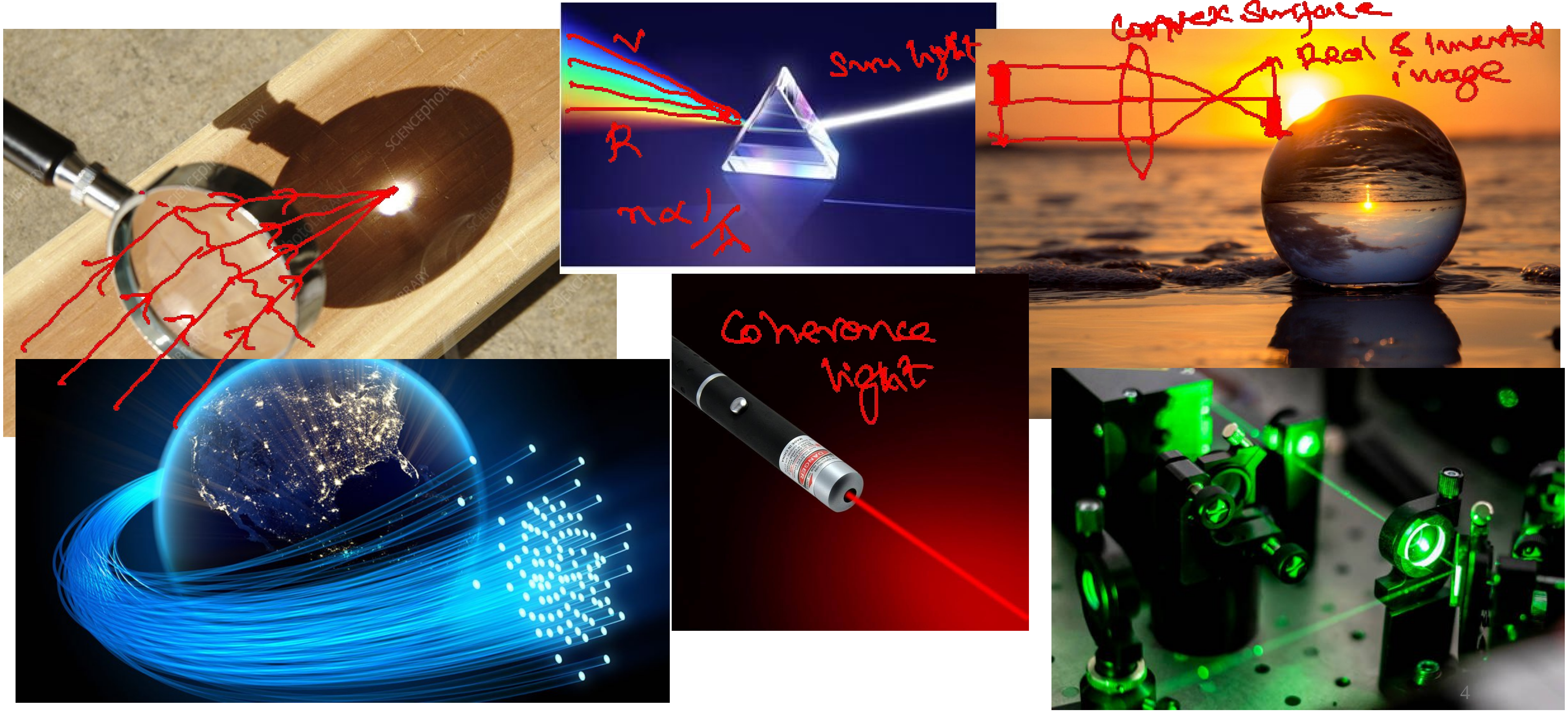
- Ajoy Ghatak, Optics, 6 ed. McGraw Hill Education (India) Pvt Ltd. 2016.
- Eugene Hecht, Optics, 4th ed. Pearson Education, Inc., 2002.
- David J Griffiths, Introduction to electrodynamics, Prentice Hall 1999.



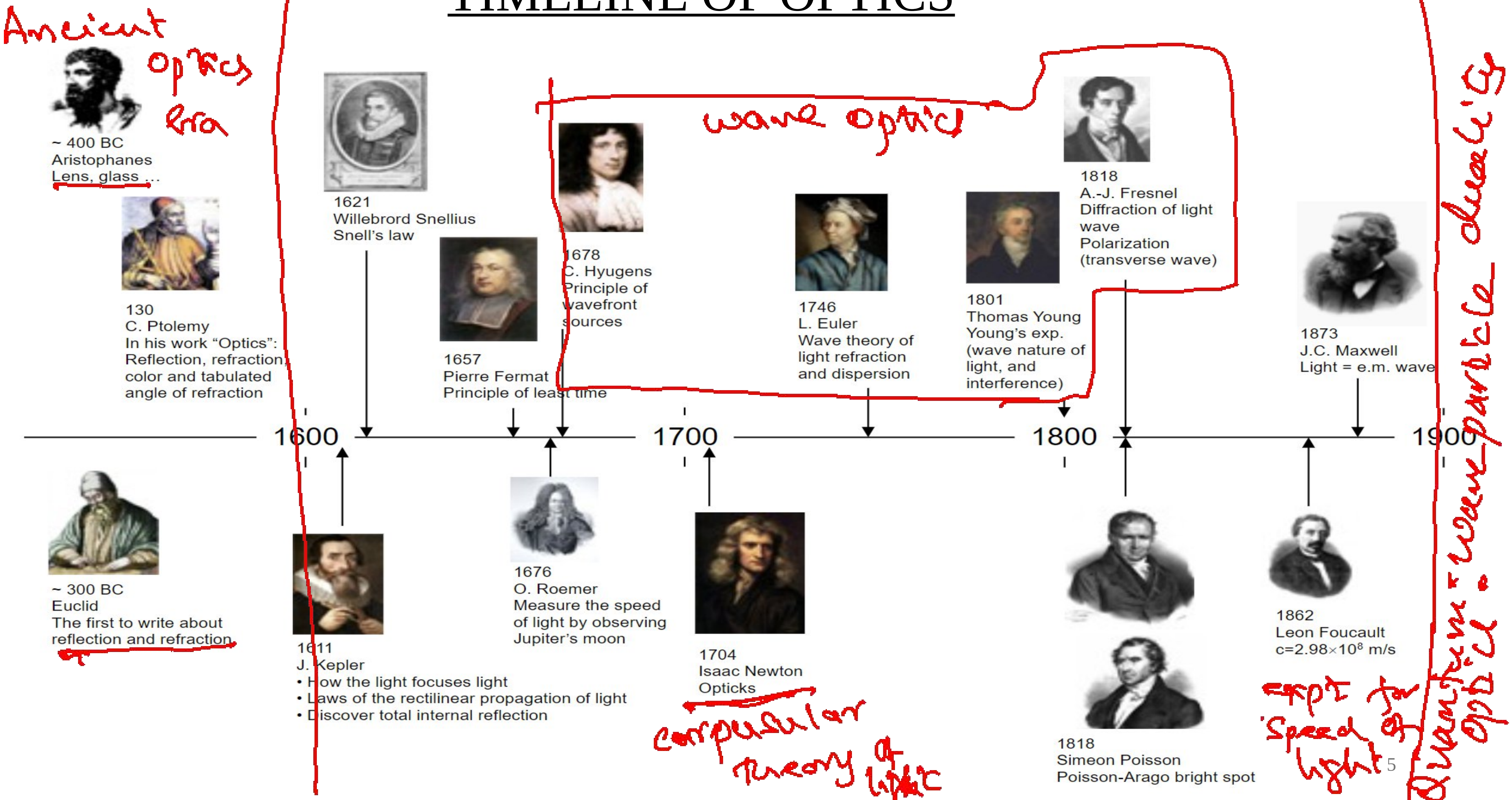


OPTICS

Optics is the branch of physics that studies the behavior and properties of light.



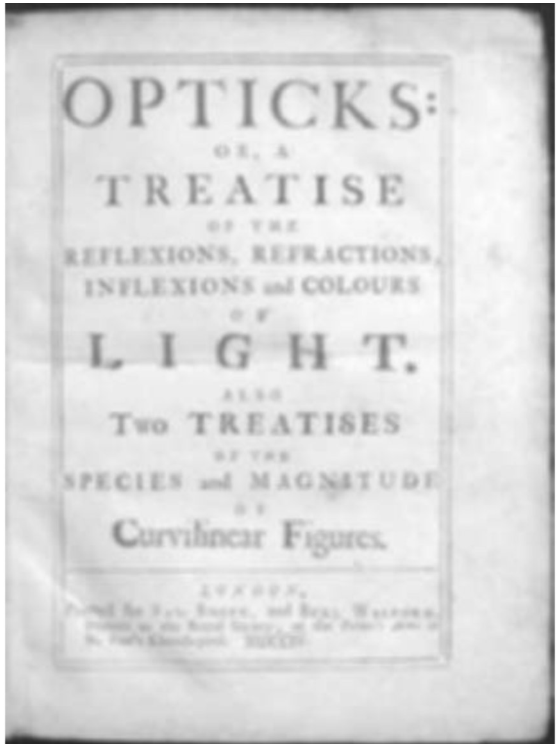
TIMELINE OF OPTICS



A BRIEF HISTORY

Issac Newton

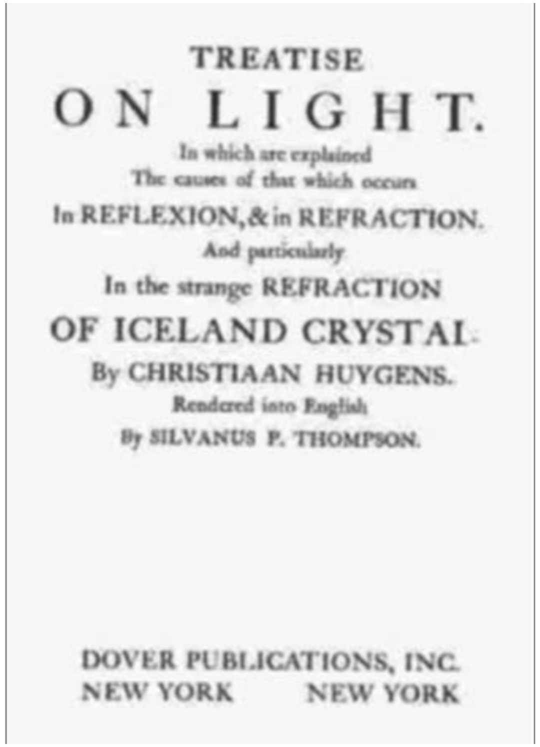
Corpuscular Theory of light



1704

Christiaan Huygens

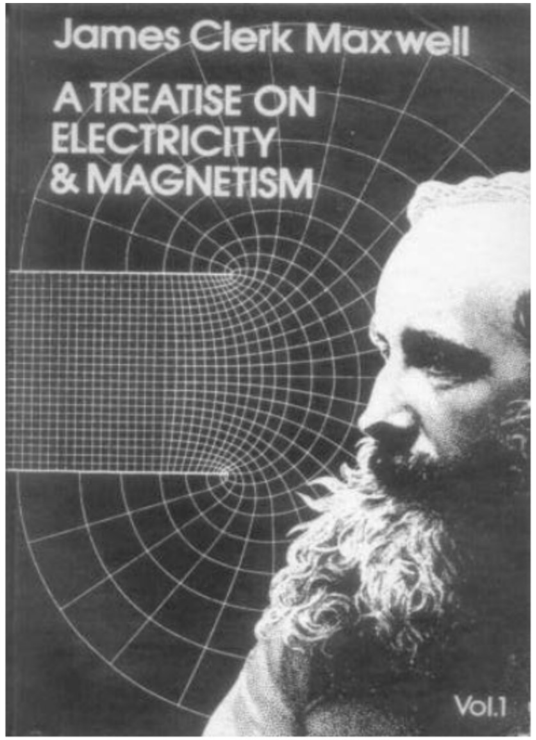
Wave Theory of light



Traite de La Lumiere (1690)

James Clerk Maxwell

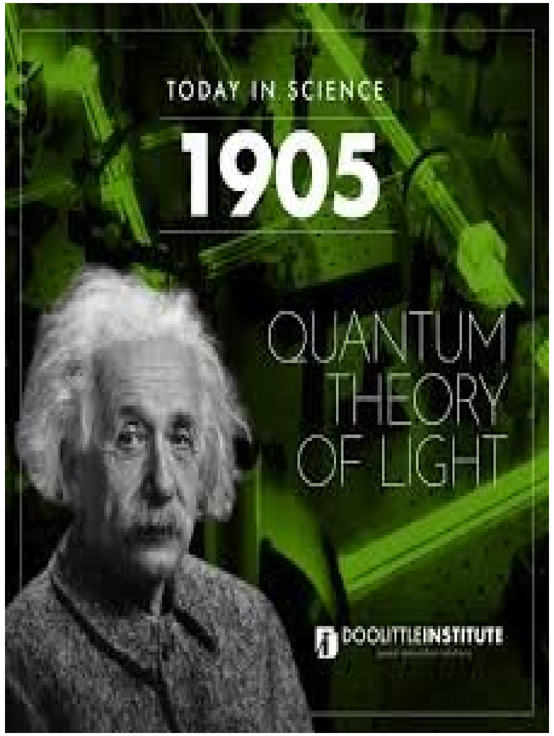
Electromagnetic Theory of light



1865

Albert Einstein

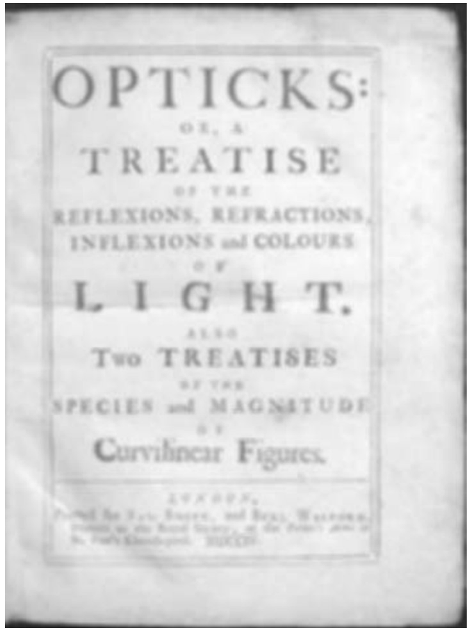
Quantum Theory of light

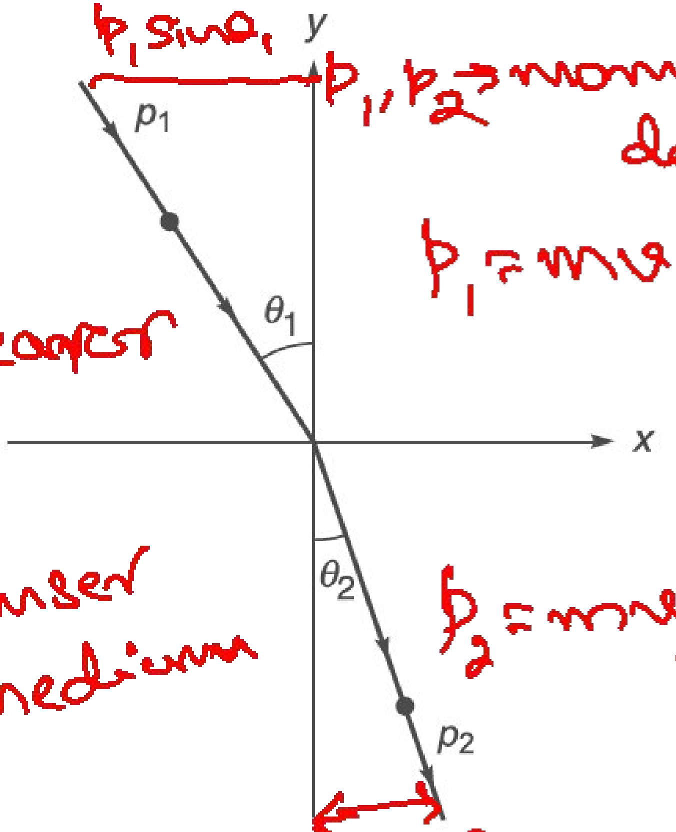


1905

THE CORPUSCULAR MODEL OF LIGHT

1600-1700s: Rene Descartes, Issac Newton (1704)





Refraction of a corpuscle

$p_1, p_2 \to$ momentum of light particle in rearer & denser medium.

$\theta_1 \to$ angle of incidence

$\theta_2 \to$ angle of refraction

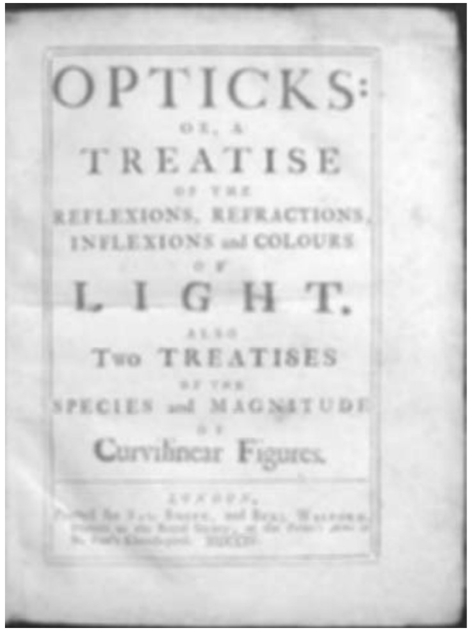
Based on conservation of linear momentum in the light propagation, we have $p_1 \sin\theta_1 = p_2 \sin\theta_2$

$$\Rightarrow \frac{p_1}{p_2} = \frac{\sin\theta_2}{\sin\theta_1} \to \text{Snell's law}$$

$$\Rightarrow \frac{v_1}{v_2} = \frac{\sin\theta_2}{\sin\theta_1} < 1 \quad (\because \theta_2 < \theta_1)$$

That contradicts that $v_1 < v_2$ (that is not true)

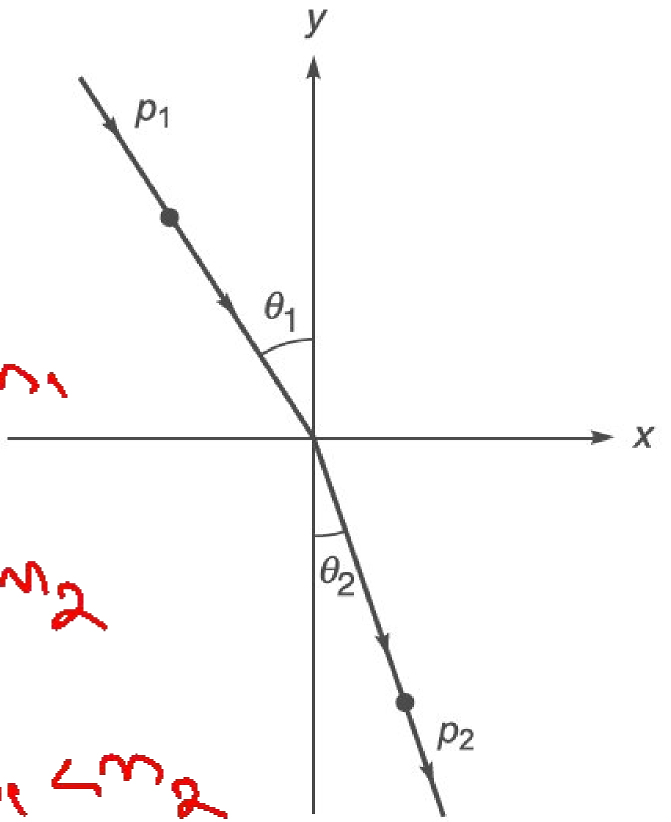
THE CORPUSCULAR MODEL OF LIGHT



1600-1700s: Rene Descartes, Issac Newton (1704)

$p_1 \sin\theta_1 = p_2 \sin\theta_2$ (conservation of momentum)

$$\boxed{\frac{p_2}{p_1} = \frac{\sin\theta_1}{\sin\theta_2} = \frac{v_2}{v_1} \quad \text{(Snell's law)}}$$



n_1

n_2

$n_1 < n_2$

Refraction of a corpuscle

Snell's law

$$\frac{n_1}{n_2} = \frac{\sin\theta_2}{\sin\theta_1}$$

- The rectilinear propagation of light which results in the formation of sharp (dark) shadows, and
- Light could propagate through vacuum.
- It is not consistent with experimental observations. It predicts that if the ray moves towards the normal (i.e., if the refraction occurs at a denser medium) its speed would become higher.

THE WAVE MODEL OF LIGHT (Hooke, Huygens (1678), Thomas Young (1802), …)

Propagation of the disturbances in time and space is termed as wave

Example: Propagation of transverse waves on a string

→ amplitude of the displacement

$$y(x,t) = a\sin\underbrace{(kx - \omega t)}_{\text{Phase of the wave}} = a \sin [k(x - vt)]$$

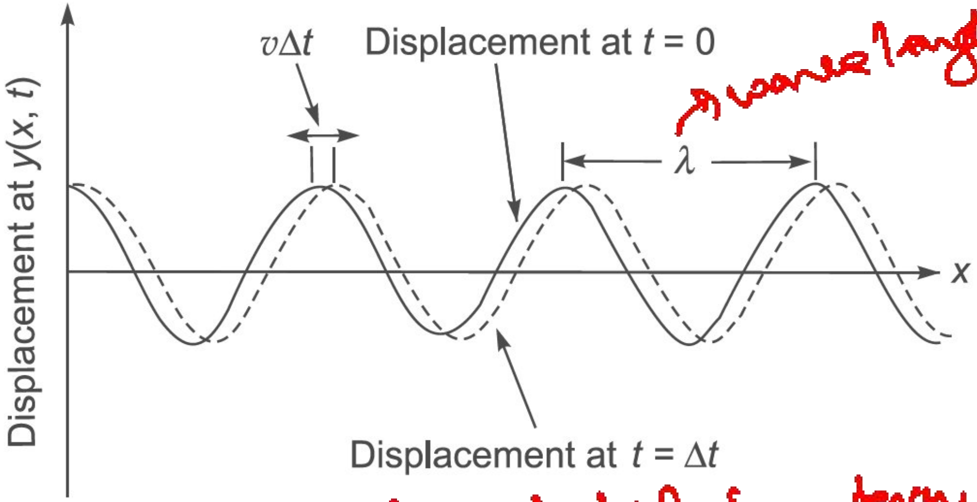
Displacement in x & t

$$v = \frac{\omega}{k} \text{ and } \omega = 2\pi\nu \;;\; \lambda = \frac{2\pi}{k}$$

k → wave number

v → phase velocity

ω → angular frequency



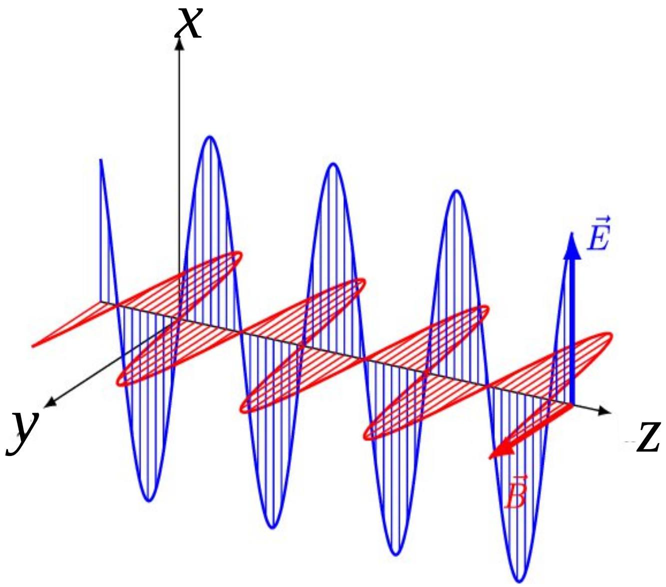
→ wavelength

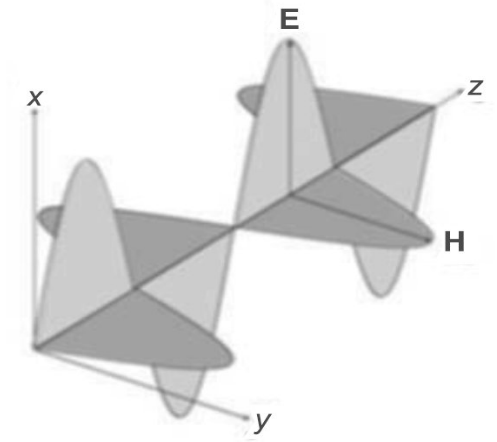
time is related in terms of phase of the propagation

The wave theory was not accepted until 1802 when Thomas Young performed the famous interference experiment which could only be explained based on a wave model of light.

1808: Polarization of light (Malus)
1816: Fresnel's diffraction experiments
1849 and 1856: Measurement of speed of light
1865: Maxwell's EM Theory
1888: Heinrich Hertz: Experimental realization/detection of EM waves

MAXWELL'S ELECTROMAGNETIC (EM) WAVES





x-polarized wave

Maxwell's theory: One of the greatest unification in Physics

Light waves are electromagnetic waves and EM waves are transverse in nature.

E, H(B) & z are ⊥r to each other

Propagation of EM waves: Plane wave solutions of Maxwell's wave equations

unit vector (polarization)

Electric field

$$\boldsymbol{E}(z,t) = \hat{x}\, E_0 \cos(kz - \omega t)$$

$$\boldsymbol{H}(z,t) = \hat{y}\, H_0 \cos(kz - \omega t)$$

magnetic field

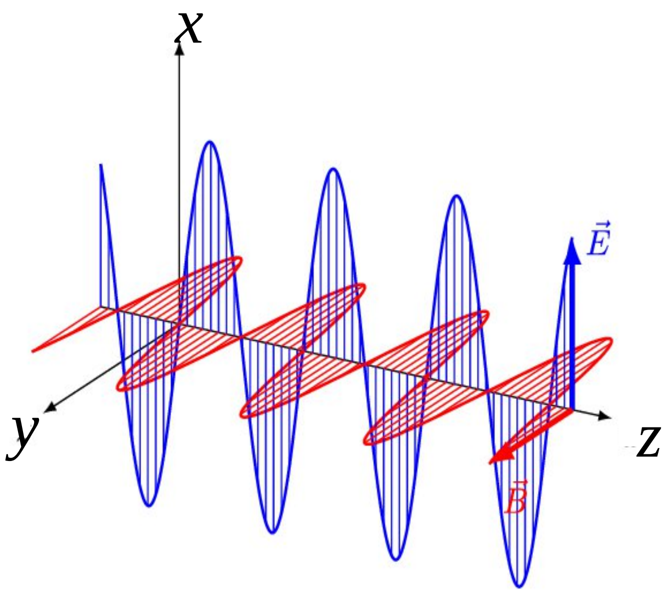
amplitude

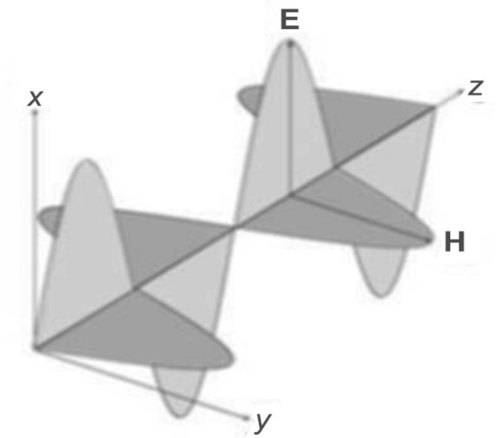
Where, $H_0 = \sqrt{\varepsilon_0/\mu_0}\; E_0$

$\varepsilon_0 = 8.86 \times 10^{-12}\ C^2N^{-1}m^{-2}$ and $\mu_0 = 12.57 \times 10^{-7}\ Ns^2C^{-2}$

$c = \dfrac{1}{\sqrt{\varepsilon_0 \mu_0}} = 3\times10^{8}\ m/s$

MAXWELL'S ELECTROMAGNETIC (EM) WAVES





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Where, $H_0 = \sqrt{\varepsilon_0/\mu_0}\; E_0$

$\varepsilon_0 = 8.86 \times 10^{-12}\ C^2 N^{-1} m^{-2}$ and $\mu_0 = 12.57 \times 10^{-7}\ Ns^2 C^{-2}$

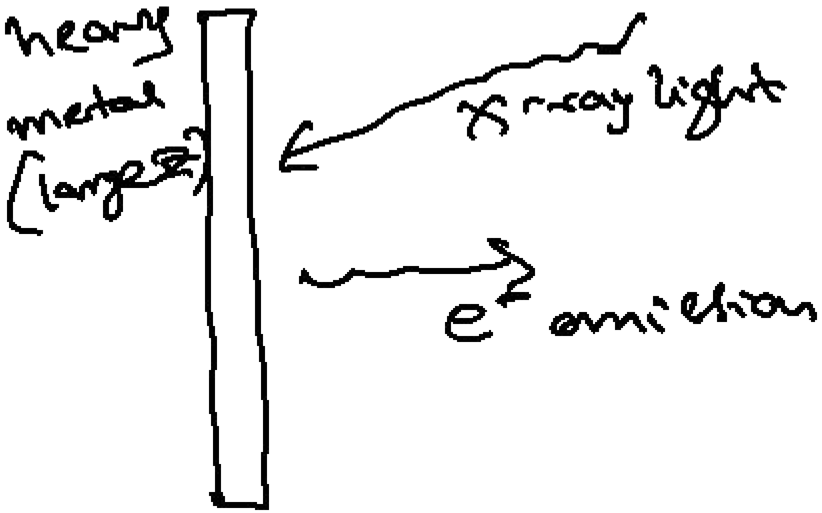
Max Planck: "*Maxwell's theory remails for all time one of the greatest triumphs of human intellectual endeavors*"

WAVE PARTICLE DUALITY

Einstein, de Broglie, Wilson, Millikan, Compton, …

The experiment on electron emission by X-rays

Two observations of the expt.



① K.E of the emitted electron remained independent of x-ray intensity

② Energy of emitted e^- increased when the frequency of x-ray increased

Acc. classical theory the larger amplitude (intensity) of the wave should result in the larger K.E of the emitted e^-, however this does not explain the above expt. observation.

$\therefore$ Einstein put forward the quantum theory of light assuming the light consists of energy quanta,

WAVE PARTICLE DUALITY

Einstein, de Broglie, Wilson, Millikan, Compton, …

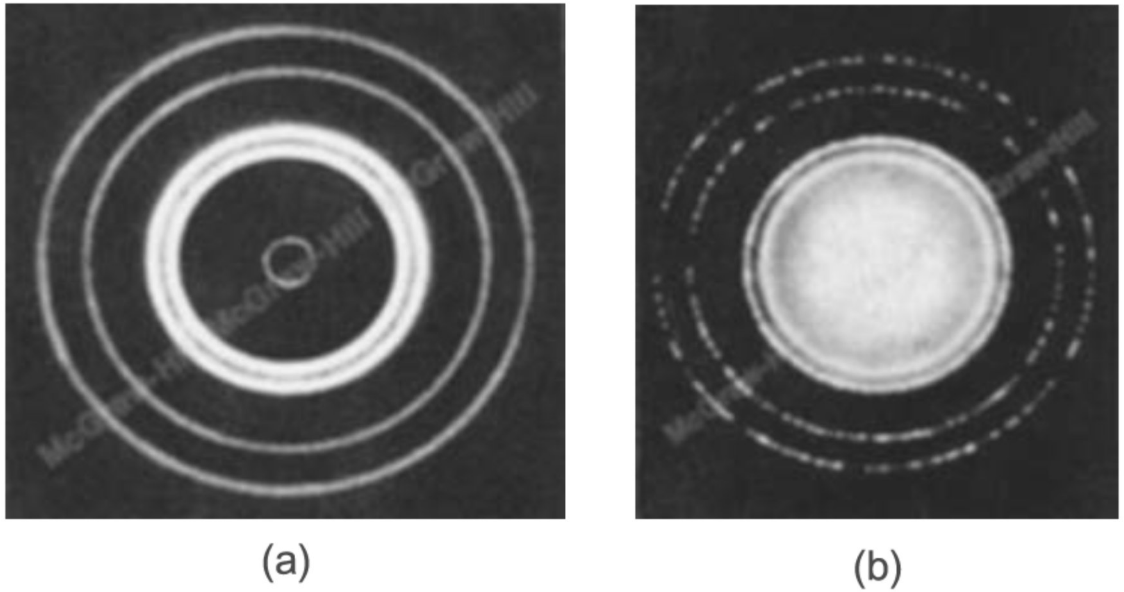
Theory of Photoelectric effect in 1905: $\boxed{\boldsymbol{E = h\upsilon}}$

momentum of light quantum $\boxed{p = \frac{h\nu}{c} = \frac{h}{\lambda}}$

Particle nature of radiation

… radiation energy consists of indivisible quanta of energy hν which are reflected undivided .. that radiation must, therefore, possess a kind of molecular structure in energy, which of course contradicts Maxwell's theory.

$\lambda_B = h/p$ → de Broglie wavelength of e^-



Diffraction patterns for X-rays and electron beam

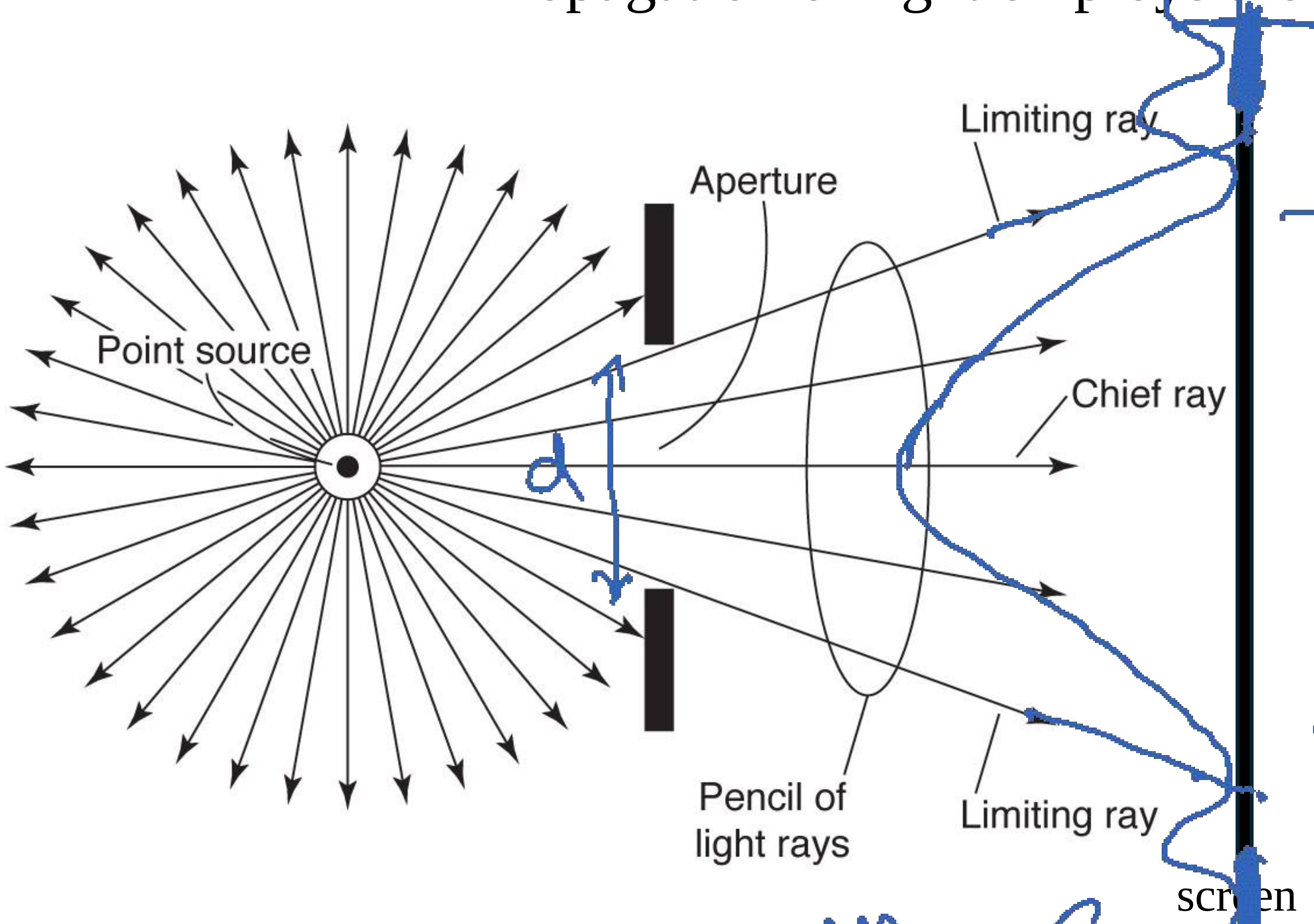
- ✓ Wave Nature of electrons developed by de Broglie
- ✓ Diffraction experiments by Wilson, Davidson and Germer, and Thomson

confirmed the wave nature of e^- in the diffraction pattern

GEOMETRICAL (RAY) OPTICS

Propagation of light employs the concept of Rays.

→ Rectilinear propagation of light.



→ Start decreasing 'd' gradually

→ When 'd' ≈ wavelength of light, we start to see diffraction patterns that ceases the light to have well defined boundaries on the screen.

→ Due to the finiteness of λ, that causes the diffraction (red curve)

→ In the $\boxed{\text{limit } \lambda \to 0}$

→ No diffraction

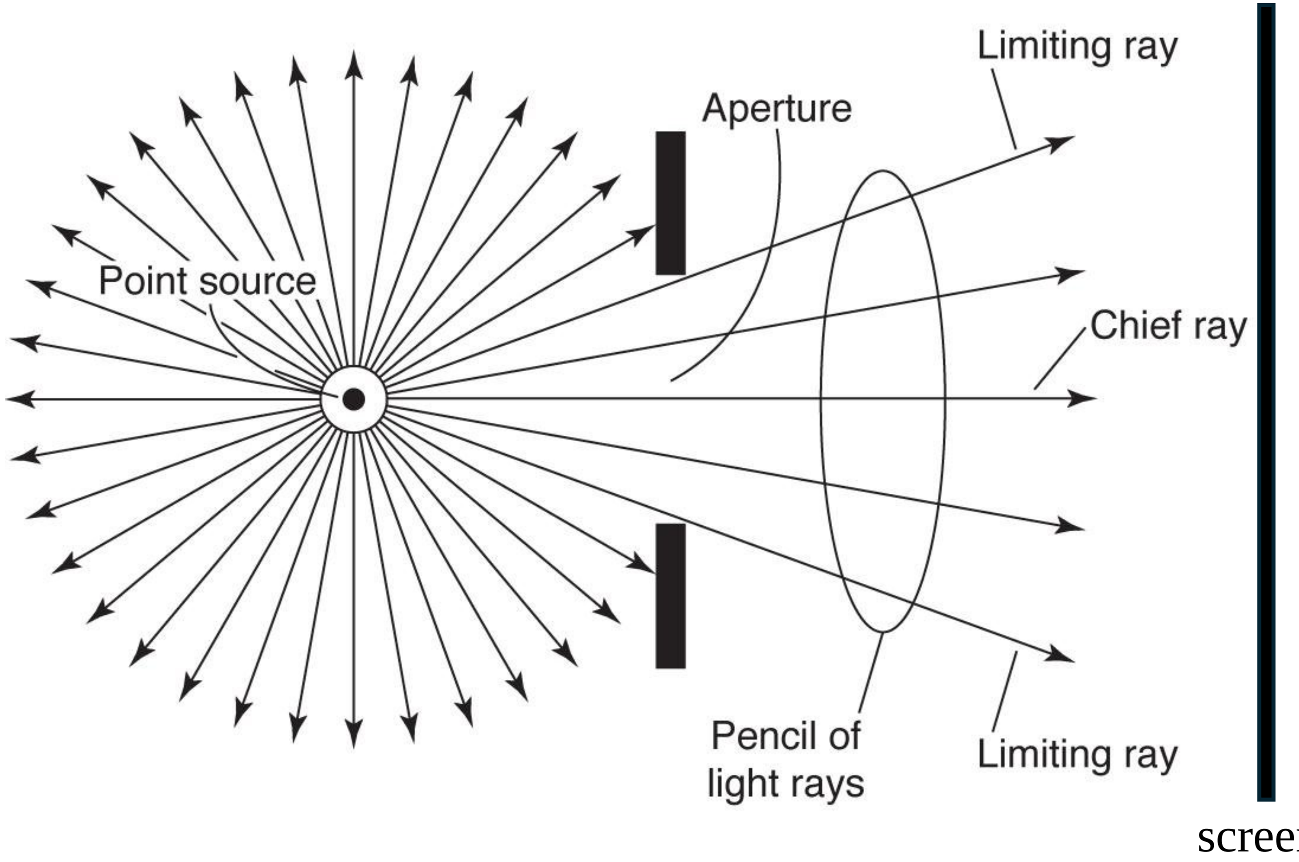
→ Definite boundary for the shadow

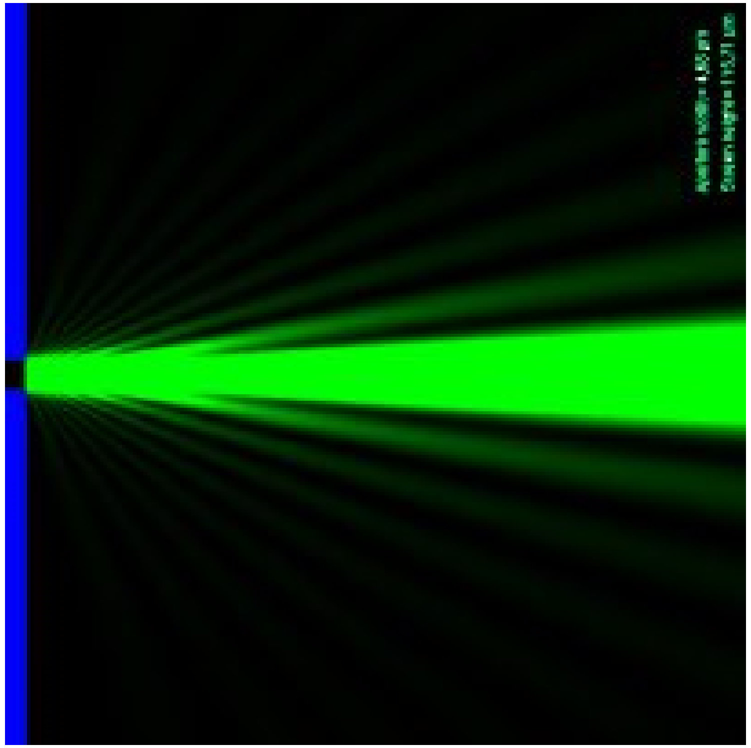
d → Aperture width ~ Cm

→ patch of light falling on the screen will show well defined boundaries.

GEOMETRICAL (RAY) OPTICS

Propagation of light employs the concept of Rays.



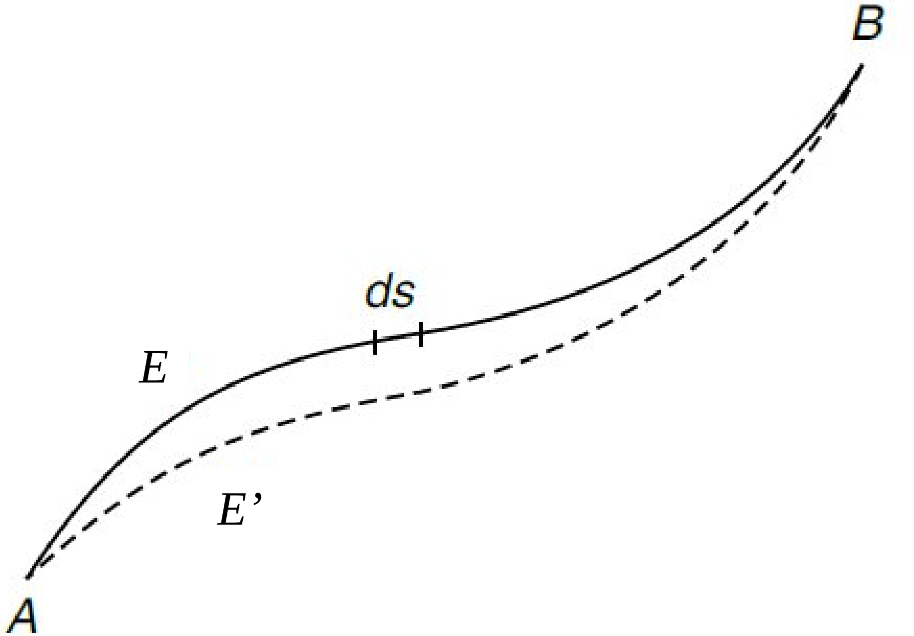


The field of optics under the approximation of zero wavelength limit $(\lambda \to 0)$ (i.e. neglecting the finiteness of the wavelength) is called Geometrical (Ray) optics.

FERMAT'S PRINCIPLE

According to this principle the ray will correspond to that path for which the **time taken is an extremum** in comparison to nearby paths.

→ light takes the least time to travel from A → B



$$\oint_A^B n\, ds = \text{Optical Path length}$$

stationary for the light wrt the variation in nearby paths

$n : n(x,y,z)$ is the position dependent Refractive index

if n is homogeneous then the path of light would be a straight line.

→ line corresponds to minimum optical path length.

$$\delta \oint_A^B n\, ds = 0$$ **Fermat's principle**

The actual ray path between two points is the one for which the **optical path length is stationary** with respect to variations of the path: **<u>Fermat's Principle</u>**

WAVE OPTICS

A *classical traveling wave is a self-sustaining disturbance of a medium, which moves through space and in time transporting energy and momentum.*

The scalar wave function is given by (ϕ)

$$\psi(x,t) = a \sin\underbrace{(kx - \omega t)}_{\text{phase of the wave}}$$

→ This wavefunction should satisfy the wave eqn.

$$\nabla^2 \psi = \frac{1}{v^2}\frac{\partial^2 \psi}{\partial t^2}$$

→ wave equation is linear

→ The wave function should be continuous at the boundary between two media

→ Light wave travels at the 3×10^8 m/s in free space.

WAVE OPTICS

A *classical traveling wave is a self-sustaining disturbance of a medium, which moves through space and in time transporting energy and momentum.*

- **Simple Harmonic Waves**

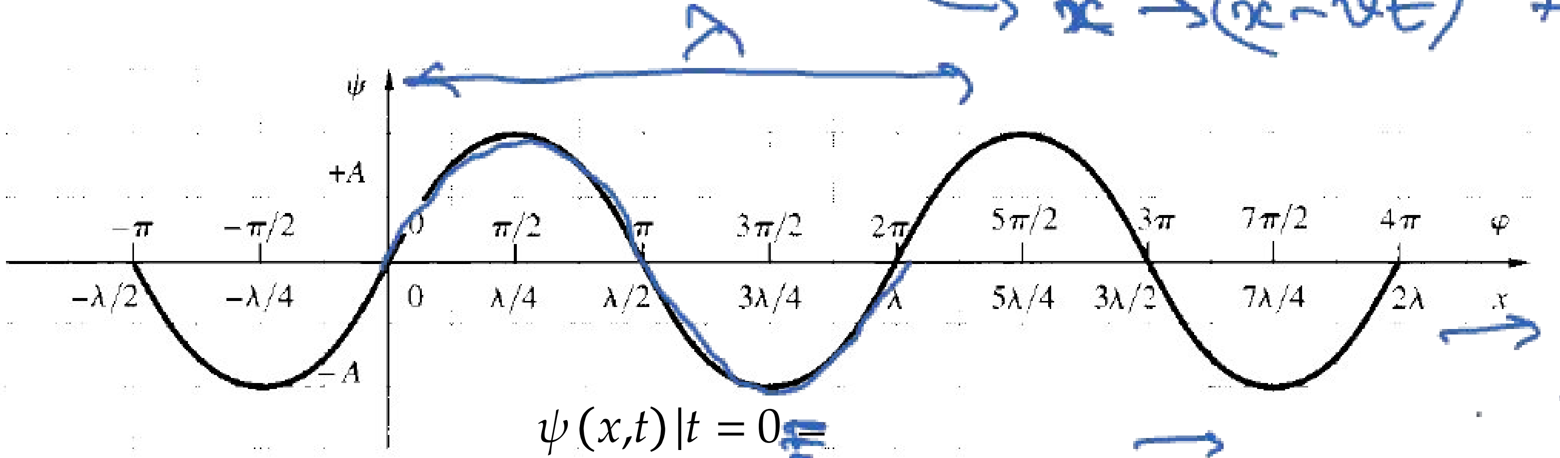
The simplest kind of periodic motion, where the displacement varies sinusoidally with time and space.

Consider any progressive harmonic wave function, $\boxed{\psi(x,t) = A \sin [\underbrace{k(x-vt)}_{\text{phase of the wave}}]}$

$x \rightarrow (x - vt)$

$k \rightarrow$ propagation number (unit of length^{-1})

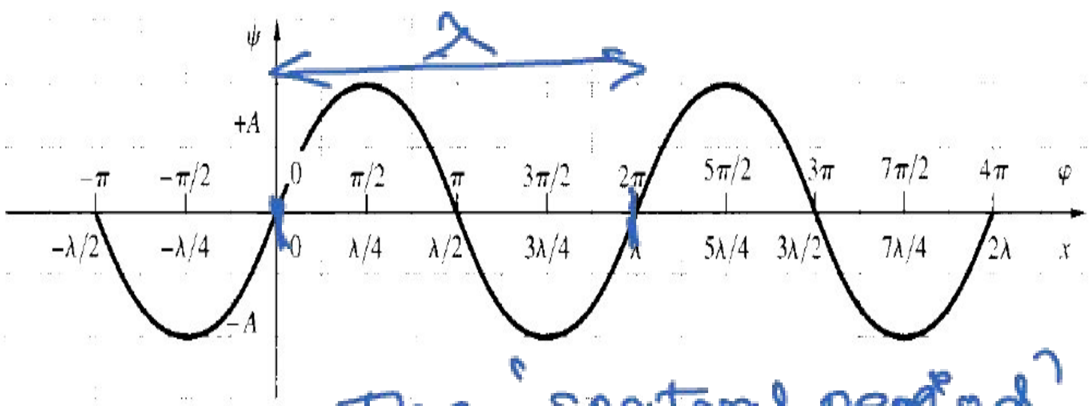
$k(x-vt) \rightarrow$ is dimension less (radians)



$\psi(x,t)|_{t=0} = A \sin (kx)$

A harmonic function, which serves as the profile of a harmonic wave.

Simple Harmonic Waves



Lets consider the wave when t is zero or constant (stationary)

then $\psi(x,t)\big|_{t=0} = A \sin kx$

Since the wave is periodic in 'x'

The 'spatial period' for the wave is given by wavelength (λ)

$\Rightarrow$ The wave is unaltered if x is changed by an amount λ, ie $x \to (x \pm \lambda)$

then $\psi(x,t)_{t=0} = \psi(x \pm \lambda, t)_{t=0}$

$= A \sin[k(x \pm \lambda)]$ → changing to radians

or $= A \sin[kx \pm 2\pi]$

$\Rightarrow$ $|k\lambda| = 2\pi$

or $\boxed{k = 2\pi/\lambda}$ → wave number or propagation number

Similarly consider the wave is stationary for x

i.e. $\psi(x,t)\big|_{x=0} = A\sin(kvt)$

& for a 'temporal period' τ, the wave is periodic and is unaltered if t is changed by an amount τ.

$$\Rightarrow \quad \psi(x,t)_{x=0} = \psi(x, t \pm \tau)$$

$$= A\sin kv(t \pm \tau)$$

$$= A\sin(kvt \pm 2\pi)$$

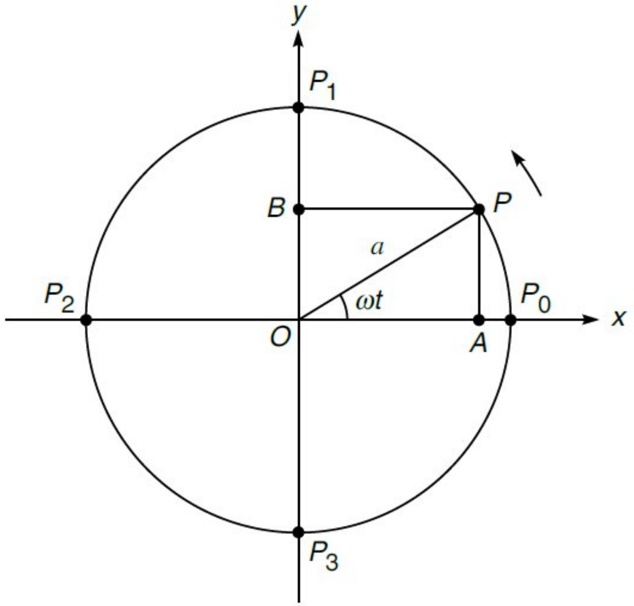
$$\Rightarrow \quad kv\tau = 2\pi$$

$$\frac{2\pi}{\lambda} v\tau = 2\pi$$

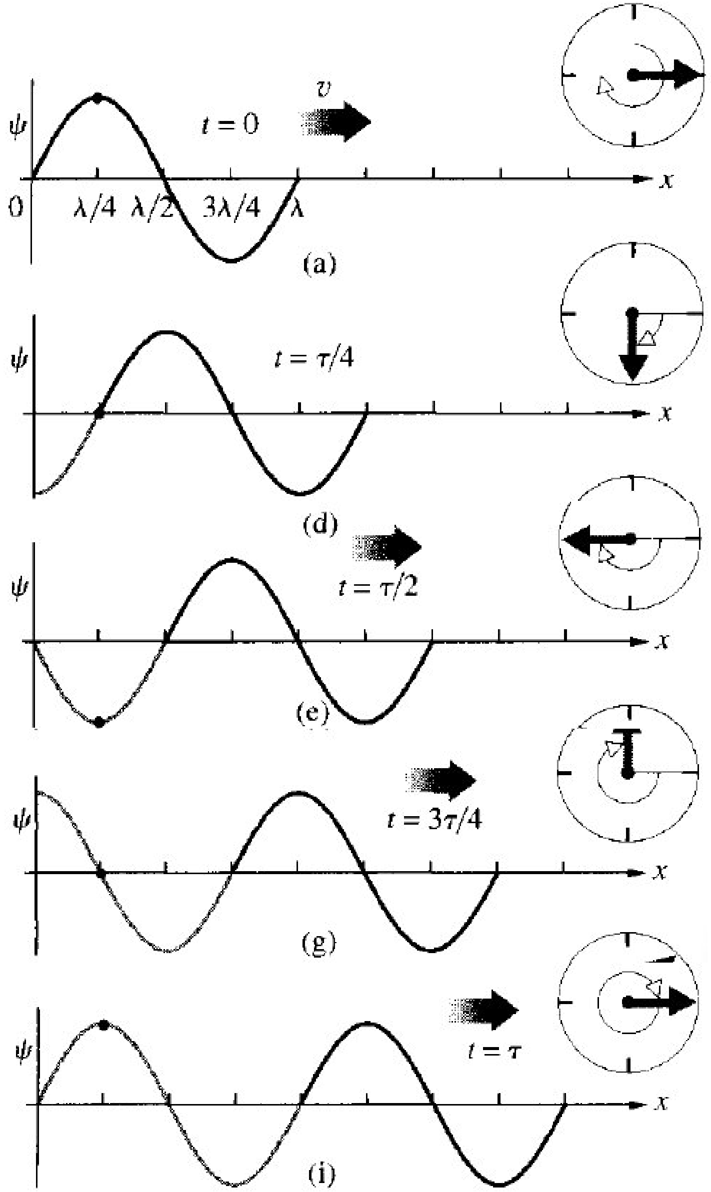
'temporal frequency'

$$v\tau = \lambda$$

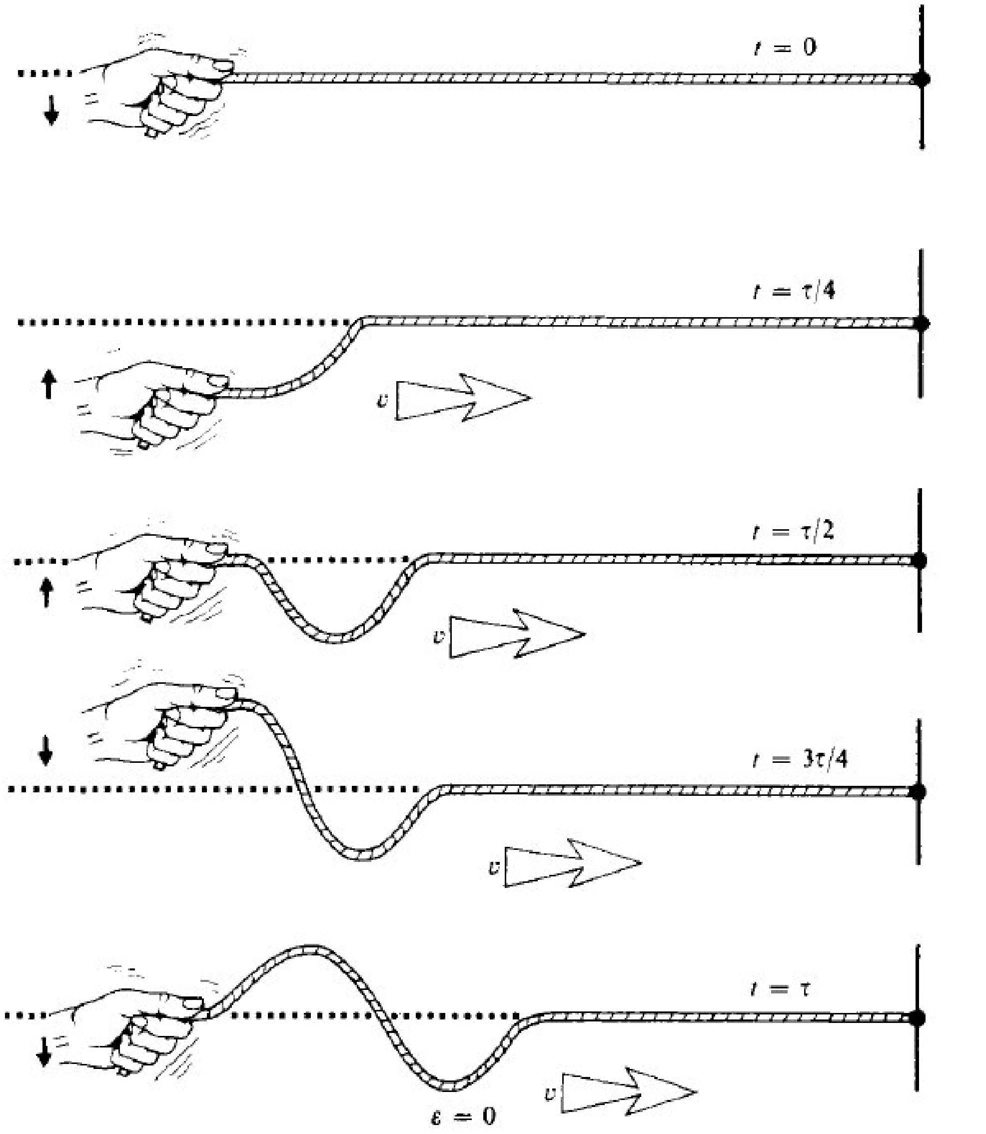
velocity of wave $\Longleftarrow$ or $\boxed{v = \lambda f}$, where $f = 1/\tau$



Simple harmonic motion



A harmonic wave moving along the x-axis during a time of one period (τ).



A harmonic wave moving along the x-axis, with $\varepsilon = 0$

PHASE VELOCITY

Total phase of the wave (ϕ)

Consider any harmonic wave function, $\psi(x,t) = a \sin\,(\underbrace{kx - \omega t + \varepsilon})$

where $\varepsilon \rightarrow$ initial phase of the wave

phase, $\phi(x,t) = (kx - \omega t + \varepsilon)$

Let's look at rate of change of ϕ wrt 't'

$$\left.\frac{\partial \phi}{\partial t}\right|_{x=c} = -\omega$$

llly

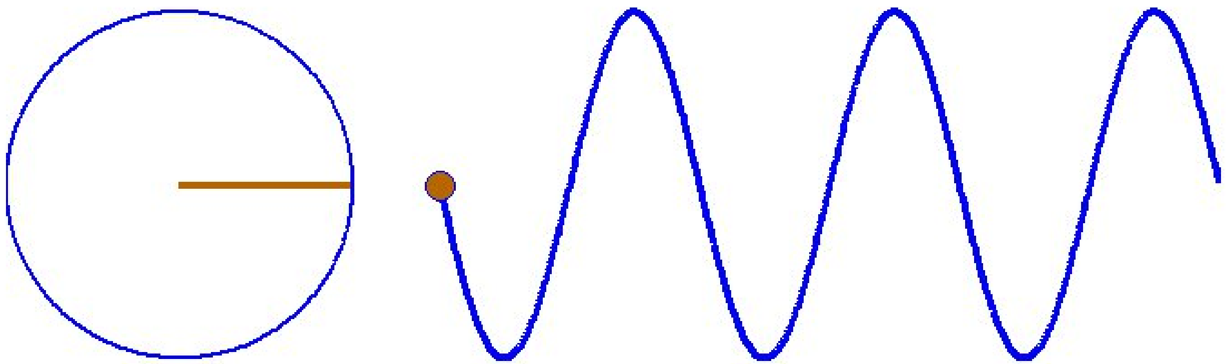
$$\left.\frac{\partial \phi}{\partial x}\right|_{t=c} = k$$

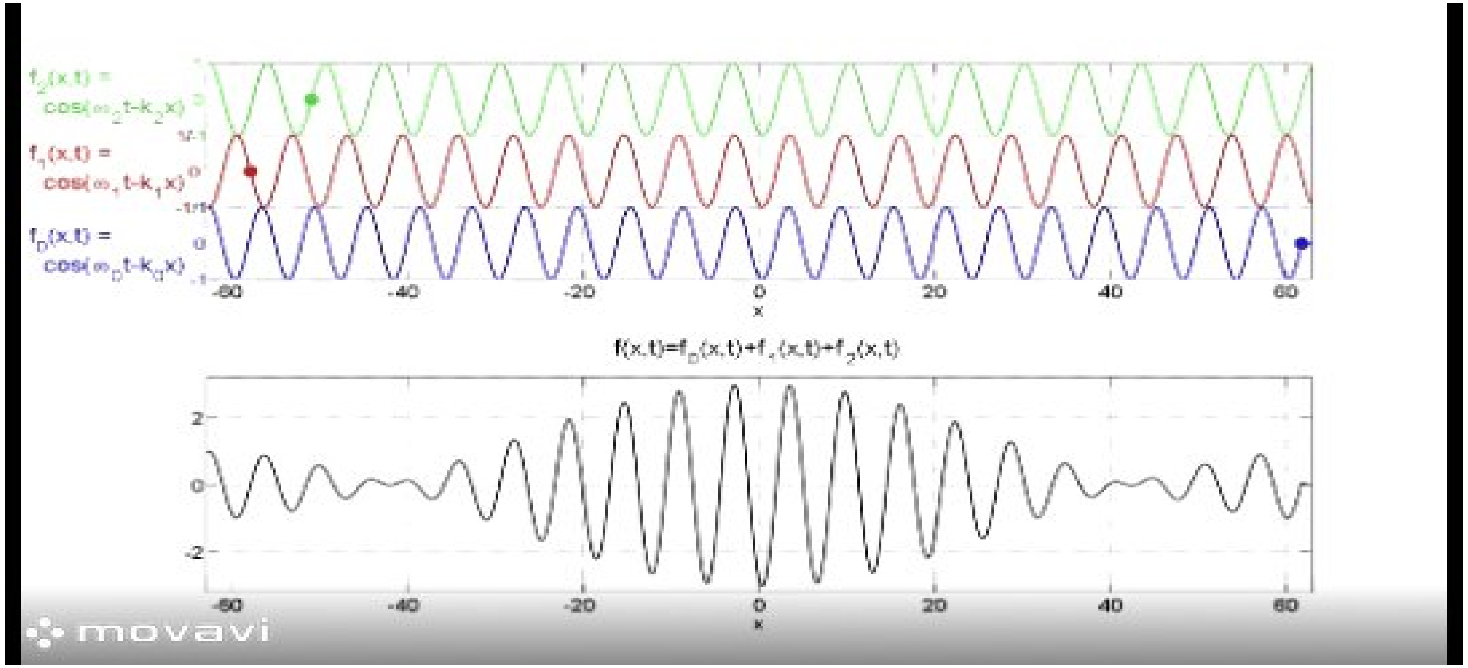
Then $v = \left.\frac{\partial x}{\partial t}\right|_{\phi} = \frac{-(\partial \phi/\partial t)_x}{(\partial \phi/\partial x)_t} = \pm \omega/k$

$$\boxed{v_p = \pm \omega/k}$$

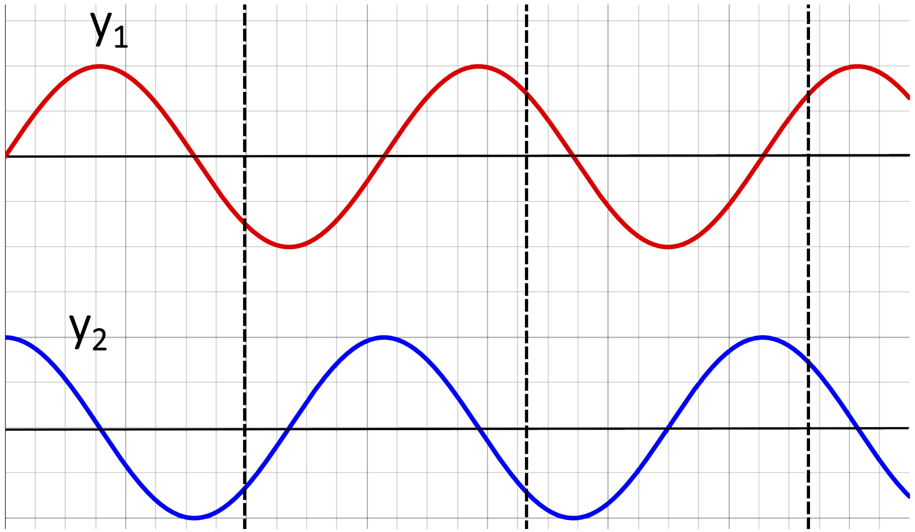
$\rightarrow$ phase velocity $\rightarrow$ speed at which the phase profile moves.

PHASE VELOCITY





SUPERPOSITION OF WAVES



Let's consider two sinusoidal waves having same frequency and same propagation direction but different initial phases ϵ_1 & ϵ_2

$$y_1(x,t) = a_1 \sin(kx - \omega t + \varepsilon_1)$$

$$y_2(x,t) = a_2 \sin(kx - \omega t + \varepsilon_2)$$

superposition principle

$$y(x,t) = y_1(x,t) + y_2(x,t)$$

$$= a_1 \sin(kx - \omega t + \epsilon_1) + a_2 \sin(kx - \omega t + \epsilon_2)$$

which can be written in the form

$$y(x,t) = a \sin(kx - \omega t + \epsilon)$$

hint [use, $\sin(a+b) = \sin a \cos b + \cos a \sin b$]

where

$$\boxed{\begin{aligned} a_1 \cos\epsilon_1 + a_2 \cos\epsilon_2 &= a \cos\epsilon \\ a_1 \sin\epsilon_1 + a_2 \sin\epsilon_2 &= a \sin\epsilon \end{aligned}}$$

$\rightarrow$ (1)

$\rightarrow$ (2)

SUPERPOSITION OF WAVES

Square and add (1) & (2)

$$a^2(\sin^2\epsilon + \cos^2\epsilon) = (a_1\cos\epsilon_1 + a_2\cos\epsilon_2)^2 + (a_1\sin\epsilon_1 + a_2\sin\epsilon_2)^2$$

$$a^2 = \left[a_1^2 + a_2^2 + 2a_1a_2\cos(\epsilon_1 - \epsilon_2)\right]$$

Resultant amplitude of superposed wave

$$\boxed{a = \left[a_1^2 + a_2^2 + 2a_1a_2\cos(\epsilon_1 - \epsilon_2)\right]^{1/2}}$$

and if we divide (2) by (1)

resultant argument

$$\boxed{\tan\epsilon = \frac{a_1\sin\epsilon_1 + a_2\sin\epsilon_2}{a_1\cos\epsilon_1 + a_2\cos\epsilon_2}}$$

SUPERPOSITION OF WAVES

if $\epsilon_1 - \epsilon_2 = 0, 2\pi, 4\pi, \ldots$

then $a = (a_1 + a_2)$

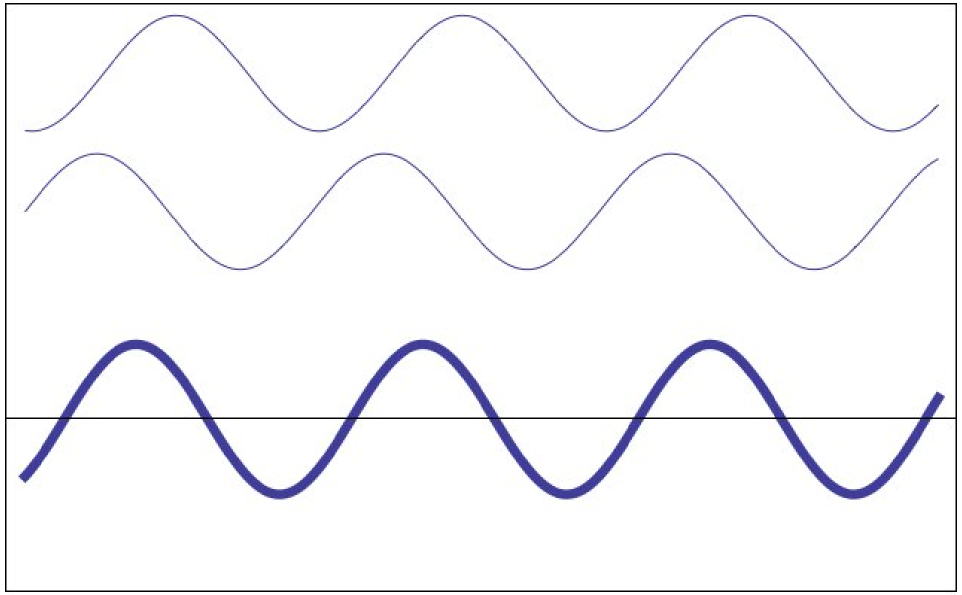
constructive interference

if $\epsilon_1 - \epsilon_2 = \pi, 3\pi, 5\pi, \ldots$

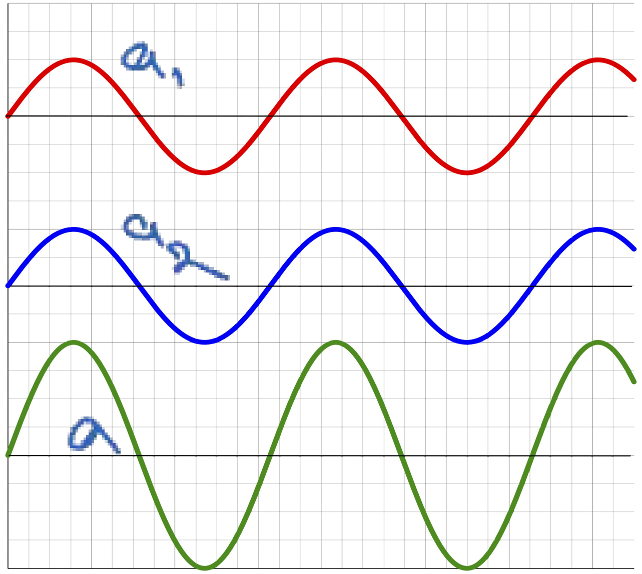
then $a = (a_1 - a_2)$

destructive interference.

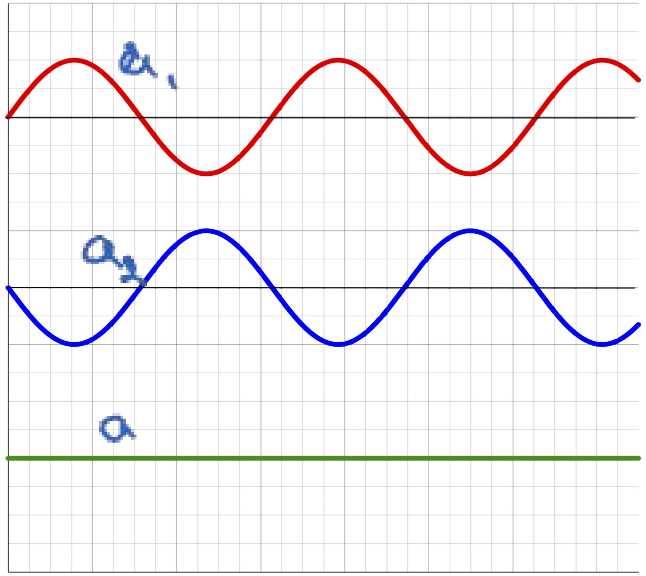
SUPERPOSITION OF WAVES



Constructive



Destructive



$a = a_1 + a_2$

$a = a_1 - a_2$

Superposition of waves $\rightarrow$ Interference phenomenon depends on the initial phase difference $(\epsilon_1 - \epsilon_2)$

RECAP

→ Phase velocity $v_p = \omega / k$

ω → angular frequency

k → wave number of wave.

→ phase velocity = wave velocity for the monochromatic wave

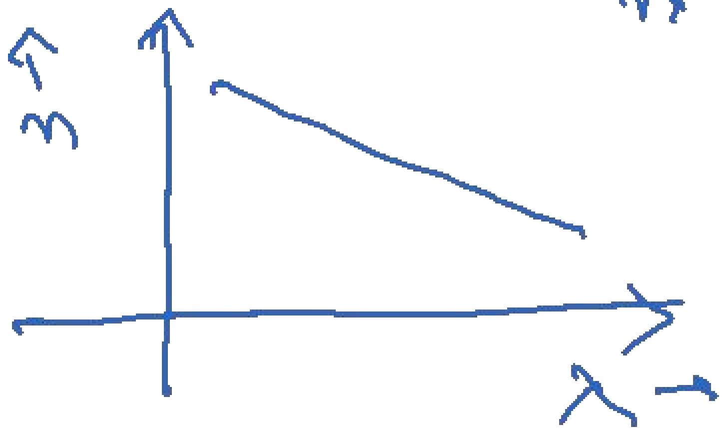
→ Phase velocity depends on the refractive index of the medium & wavelength of light

$v = \frac{c}{n}$ & $v = f\lambda$

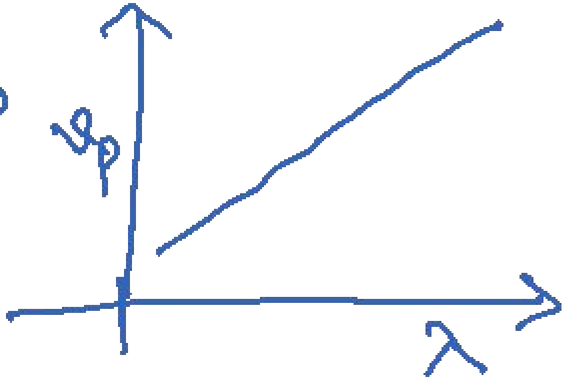
$\Rightarrow \frac{c}{n} = f\lambda \Rightarrow \frac{c}{f} = n\lambda$

$\Rightarrow \lambda_0 = n\lambda$

$$\boxed{n = \frac{\lambda_0}{\lambda}}$$

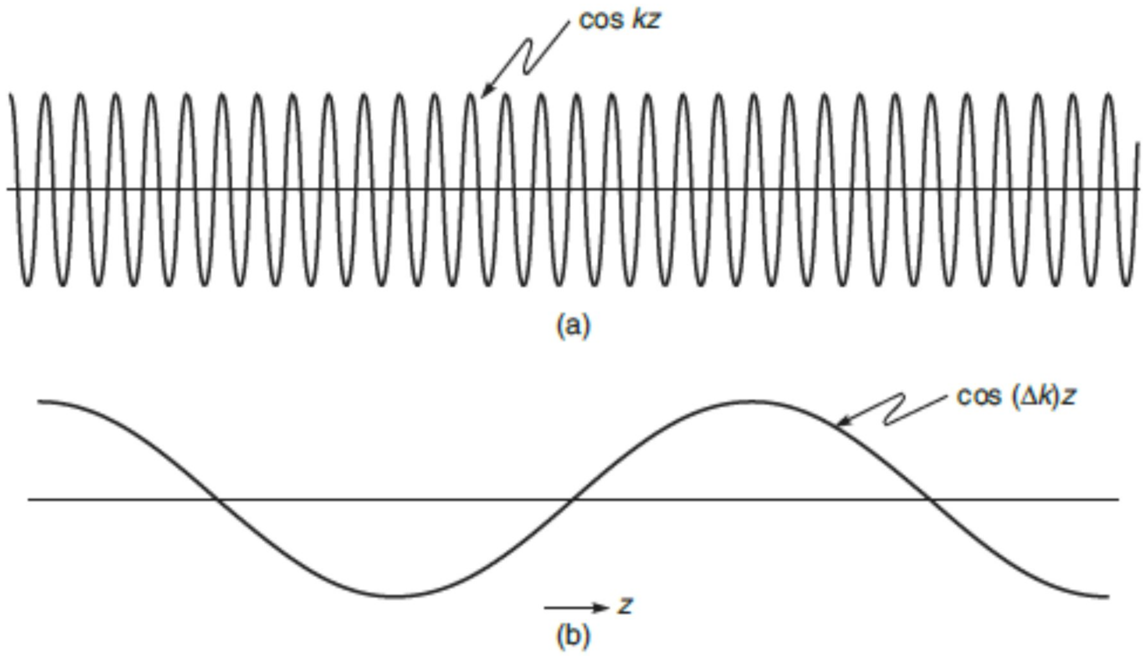


⇒



GROUP VELOCITY

→ velocity of a wave packet (collection of waves)



Variation of rapidly varying term (a) and slowly varying envelope (b) of the wave at t = 0.

→ Simple harmonic wave with same (constant) frequency & k

$$\psi(x,t) = a\cos(kz - \omega t)$$

(monochromatic / SH wave)

→ superposition of the harmonic waves with different frequency ($\Delta\omega$) and wave vector (Δk).

→ forms an envelope of waves

$$\psi'(x,t) = a\cos(\Delta k z - \Delta\omega t)$$

Taking $d\phi = 0$

$$\Delta k\, dz = \Delta\omega\, dt$$

$$v_g = \frac{dz}{dt} = \frac{\Delta\omega}{\Delta k}$$

velocity of this wave envelope $\psi'(x,t)$ is $\boxed{v_g = \frac{\Delta\omega}{\Delta k}}$

Group velocity of a wave packet (wave envelope)

Ref: Ajoy Ghatak, Optics

Superposition of two wave packets

Lets consider two wave packets with same amplitude and slightly different frequencies $(\omega+\Delta\omega)$ & $(\omega-\Delta\omega)$ and corresponding $(k+\Delta k)$ & $(k-\Delta k)$, propagating along z-direction

$$\psi_1(z,t) = A\cos\left[(k+\Delta k)z-(\omega+\Delta\omega)t\right]$$

$$\psi_2(z,t) = A\cos\left[(k-\Delta k)z-(\omega-\Delta\omega)t\right]$$

same initial phase $\epsilon = 0$

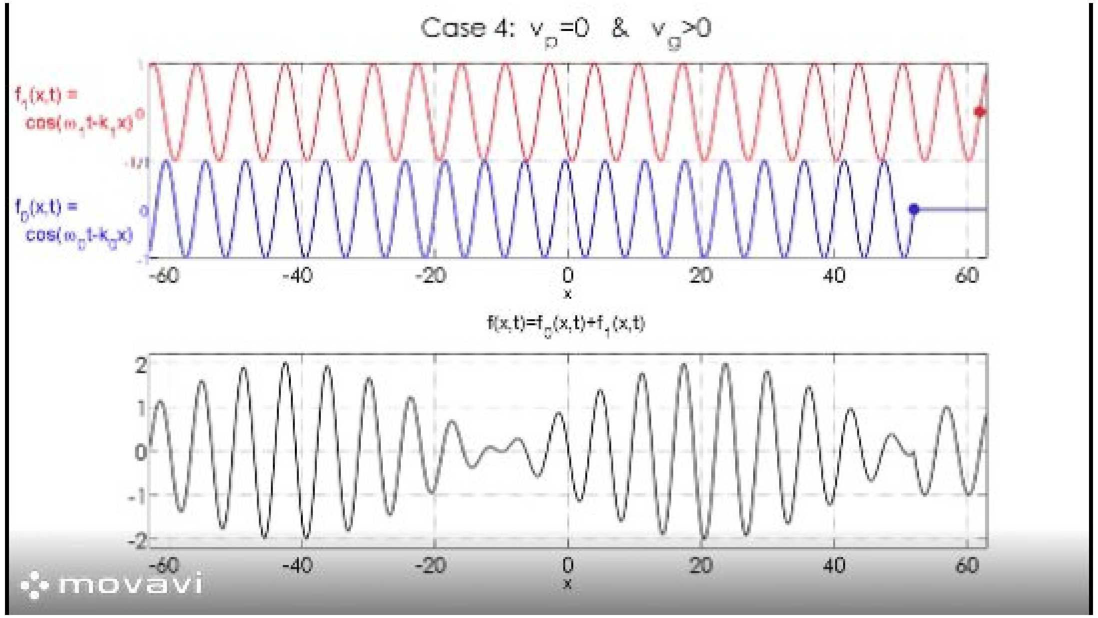
Superposition of these two waves gives

$$\psi(z,t) = 2A\,\underbrace{\cos(kz-\omega t)}_{\text{fast varying wave}}\;\underbrace{\cos(\Delta k z-\Delta\omega t)}_{\text{slowly varying wave envelop}}$$

Fast varying wave ↓ moves with the velocity $\boxed{v_p = \omega/k}$ Phase velocity

Slowly varying wave envelop ↓ moves with the velocity $\boxed{v_g = \frac{\Delta\omega}{\Delta k}}$ group velocity

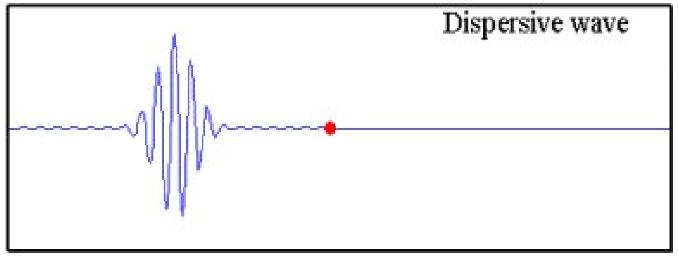
GROUP VELOCITY AND DISPERSION



Dispersion :- Dependency of the Displacement field of a propagating wave on frequency / wavelength of wave

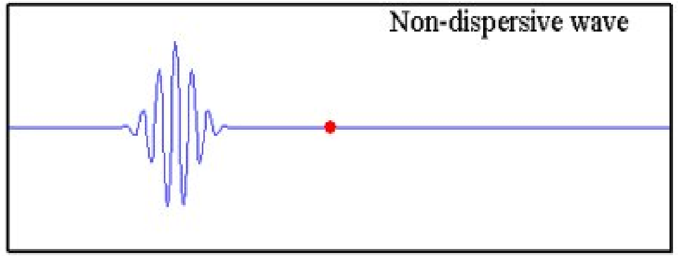
$$D(\omega) = \epsilon(\omega)\, E(\omega)$$

→ Refractive index then becomes a function of frequency / wavelength of light.



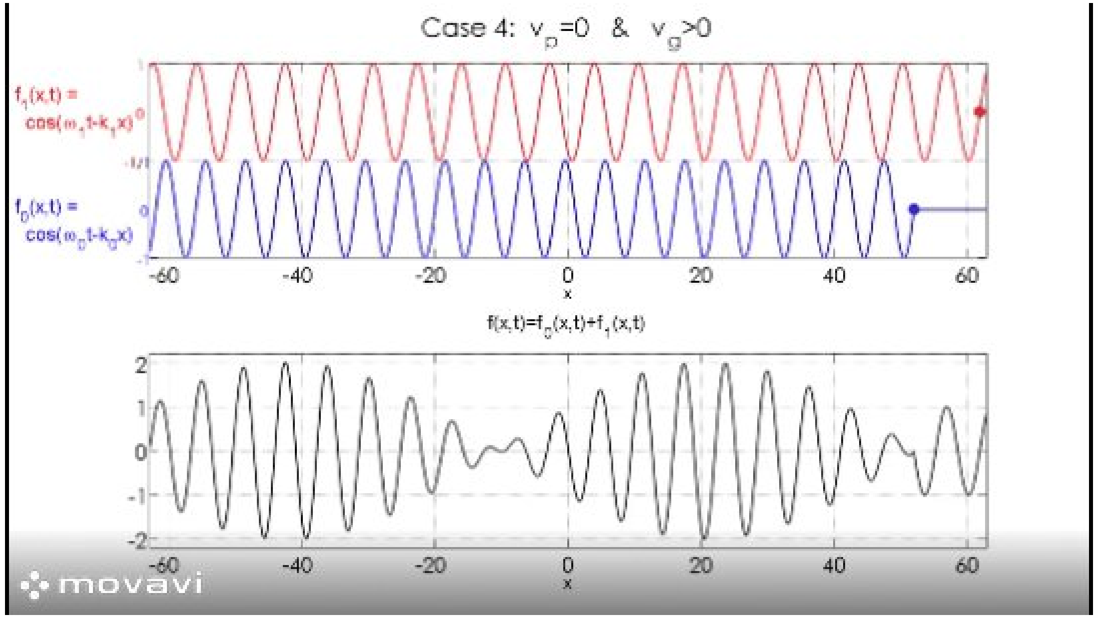
Two types of dispersion

$\frac{dn}{d\omega} > 0$; $\frac{dn}{d\lambda} < 0$ → Normal dispersion

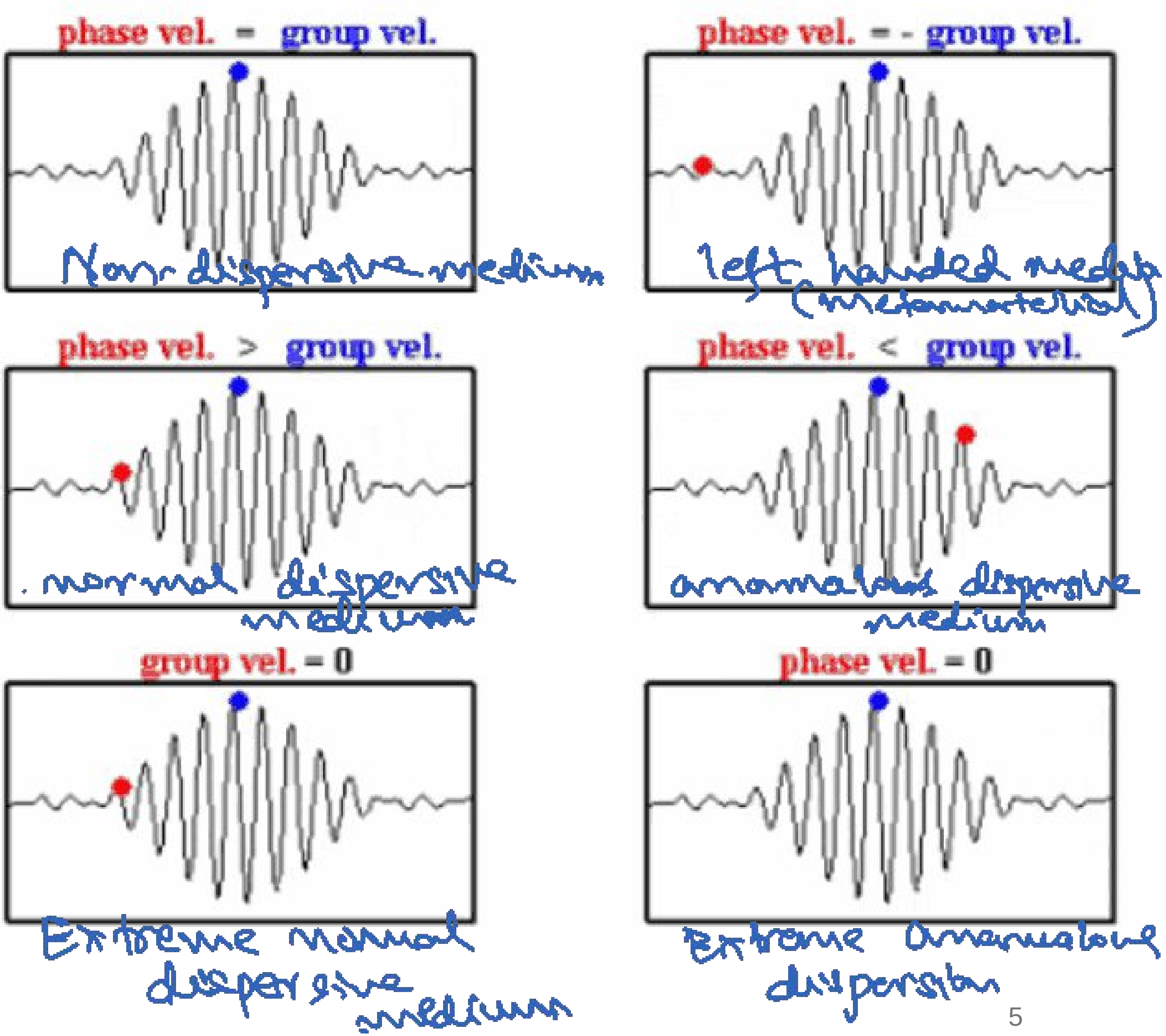


$\frac{dn}{d\omega} < 0$; $\frac{dn}{d\lambda} > 0$ → Anomalous dispersion.

GROUP VELOCITY AND DISPERSION



Different regimes of phase and group velocities in different media.



https://web.bryanston.co.uk/physics/Applets/Wave%20animations/Sound%20waves/Dispersive%20waves.htm

GROUP VELOCITY AND GROUP INDEX

Consider the wave propagating in a medium characterized by the varying refractive index $n(\omega)$

then $k(\omega) = \frac{\omega}{c} n(\omega)$

$v_p = \omega/k$

$\frac{c}{n} = \omega/k$

$k = \frac{\omega}{c} n$

then $\frac{1}{v_g} = \frac{dk}{d\omega} = \frac{1}{c}\left[n(\omega) + \omega \frac{dn}{d\omega}\right]$

$v_g = \frac{d\omega}{dk} \Rightarrow$

if in free space $n(\omega) = 1 \Rightarrow \boxed{v_g = v_p = c}$

$\omega = 2\pi f_0 = \frac{2\pi c}{\lambda_0}$ $\therefore$ $\frac{dn}{d\omega} = \frac{dn}{d\lambda_0}\frac{d\lambda_0}{d\omega}$

$= -\frac{\lambda_0^2}{2\pi c}\frac{dn}{d\lambda_0}$

Or

then $$\boxed{\frac{1}{v_g} = \frac{1}{c}\left[n(\lambda_0) - \lambda_0 \frac{dn}{d\lambda_0}\right]}$$

$dn/d\lambda_0 \rightarrow$ dispersion

In a non dispersive medium $v_g = v_p$

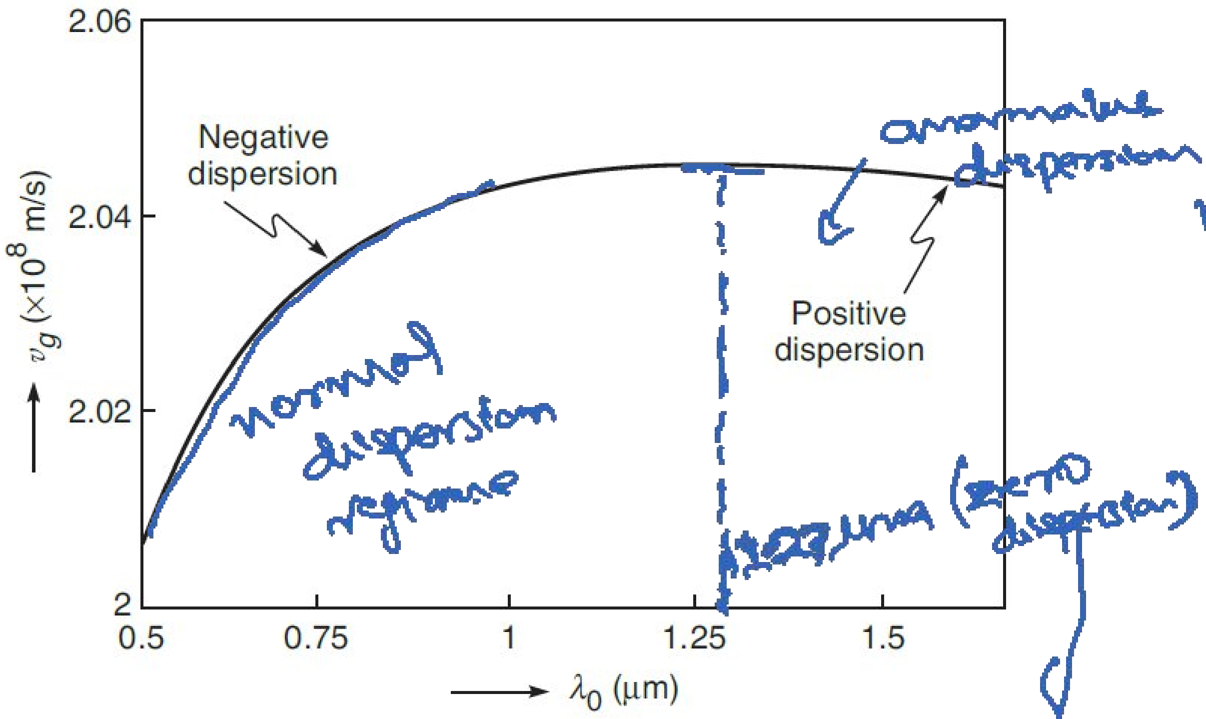
GROUP VELOCITY AND GROUP INDEX

Ref: Ajoy Ghatak, Optics

group index

$$n_g = c/v_g = \left[n(\lambda_0) - \lambda_0 \frac{dn}{d\lambda_0}\right]$$

$$v_g = c\left[n(\lambda_0) - \lambda_0 \frac{dn}{d\lambda_0}\right]^{-1}$$



Variation of group velocity with wavelength for pure silica ← Fibre core

when $\frac{dn}{d\lambda_0} > 0 \Rightarrow$ positive dispersion (anomalous)

then v_g decreases

$dn/d\lambda_0 < 0 \rightarrow$ negative (normal) dispersion

then v_g increases

For silica the group velocity becomes maximum at 1.27 μm

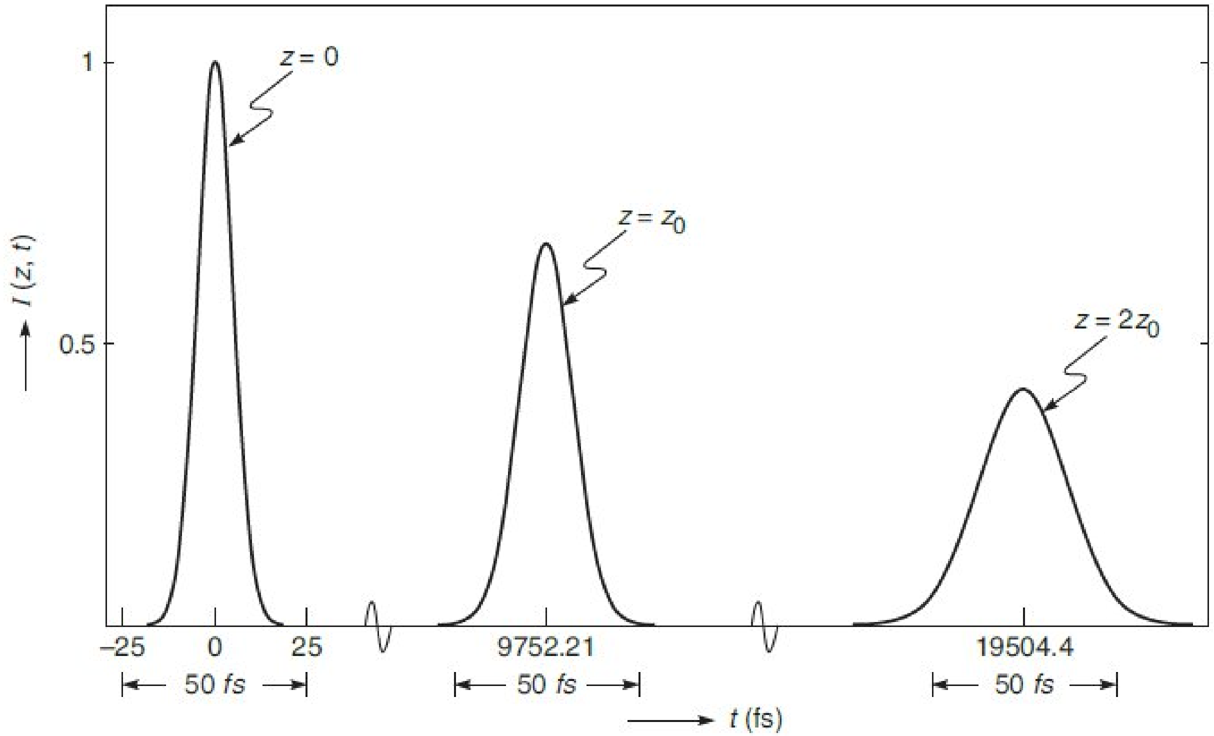
⇒ less dispersion called

∴ Good for communication applications

Dispersive medium!:

the light with different wavelength travels with different group velocity

PULSE BROADENING (DISPERSION)



Time variation of pulse intensity at different values of z: Temporal broadening of the pulse in the positive dispersion regime of silica

If you consider a light pulse, that will have certain wavelength spread ($\Delta\lambda$), since the each component of light travels with different group velocity, it will in general result in the broadening of the pulses.

The time taken by the pulse to traverse distance L of the dispersive medium is given by,

$$\tau = \frac{L}{v_g} = \frac{L}{c}\left[n(\lambda_0) - \lambda_0 \frac{dn}{d\lambda_0}\right]$$

Pulse broadening $\Delta\tau_m = \frac{d\tau}{d\lambda_0}\Delta\lambda_0$

$\Delta\tau_m \rightarrow$ material dispersion

$$\delta\tau_m = \frac{-L\,\Delta\lambda_0}{\lambda_0 c}\left[\lambda_0^2 \frac{d^2 n}{d\lambda_0^2}\right]$$

this depends on the material property.

→ pulse compresses when propagating in a negative dispersion regime
(pulse takes longer time to travel in the medium)

→ pulse broadens when propagating in a positive dispersion regime
(pulse takes less time to traverse in the medium)

GROUP VELOCITY DISPERSION (GVD)

we have pulse broadening given by

$$\Delta\tau_m = -\frac{L\,\Delta\lambda_0}{\lambda_0 c}\left[\lambda_0^2 \frac{d^2n}{d\lambda_0^2}\right]$$

$$= L\,D_m\,\Delta\lambda_0$$

where $\boxed{D_m = -\frac{\lambda_0}{c}\frac{d^2n}{d\lambda_0^2}}$ → material dispersion coefficient

and $d^2n/d\lambda_0^2$ → Group velocity dispersion.

D_m → has the dimensions ps/km.nm

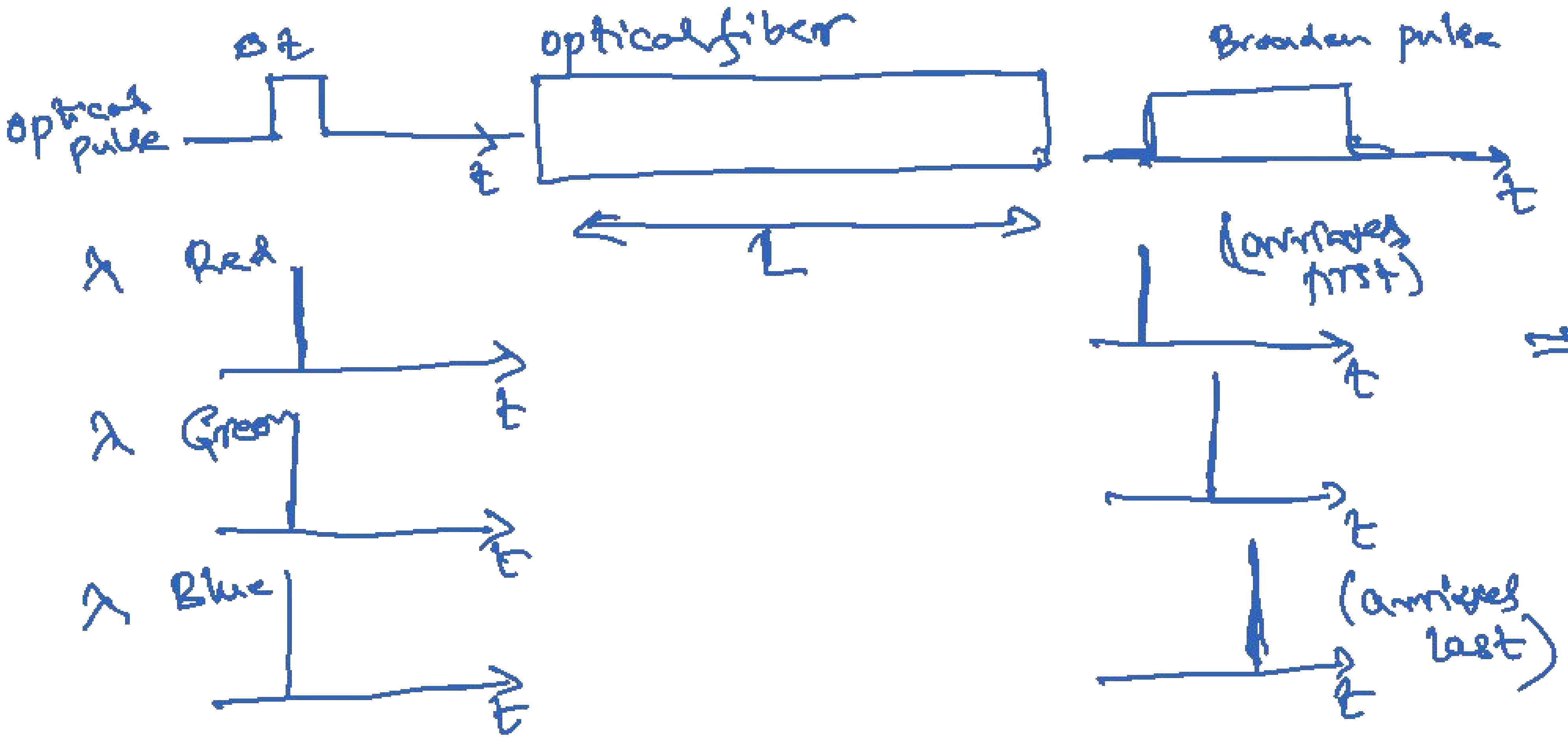
when, $D_m > 0 \Rightarrow d^2n/d\lambda^2 < 0$ → pulse broadens (anamalous regime)

→ longer λ travels faster than shorter λ

$D_m < 0 \Rightarrow \frac{d^2n}{d\lambda^2} > 0$ → pulse compresses (normal dispersion)

→ shorter λ travels faster than longer λ

PROPAGATION OF THE PULSE IN OPTICAL FIBER

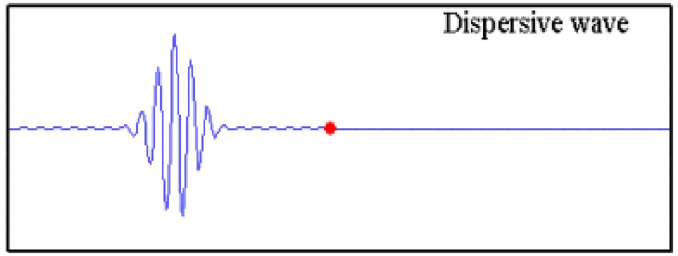


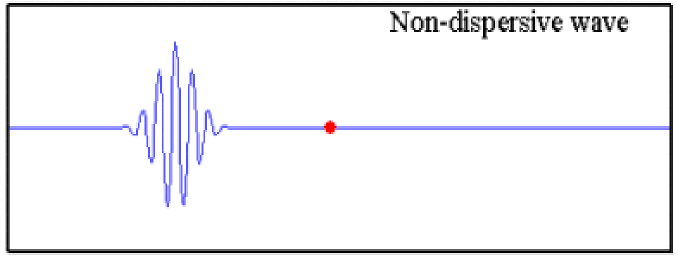
$n \propto 1/\lambda$

⇒ longer wavelength will have higher velocity and arrive first

→ Consequence is we need to increase the time seperation b/w the input pulses (causes reduced data rate) to avoid the overlapping of pulses due to dispersion.

GROUP VELOCITY AND DISPERSION





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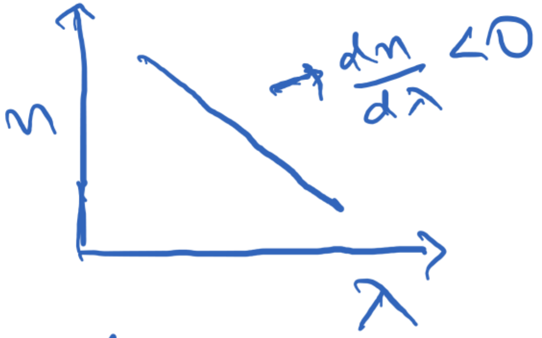
$$D(\omega) = \epsilon(\omega)\, E(\omega)$$

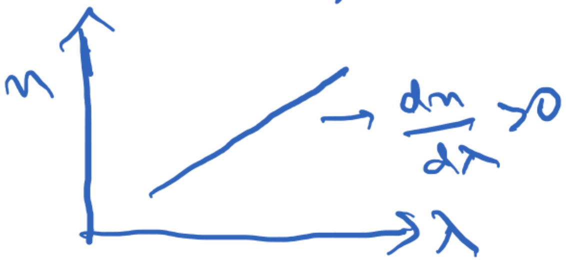
→ Refractive index the becomes a function of frequency / wavelength of light.



Two types of dispersion

$\frac{dn}{d\omega} > 0$; $\frac{dn}{d\lambda} < 0$ → Normal dispersion

$\frac{dn}{d\omega} < 0$; $\frac{dn}{d\lambda} > 0$ → Anamalus dispersion.



GROUP VELOCITY AND GROUP INDEX

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$v_p = \omega / k$

$\frac{c}{n} = \omega / k$

$k = \frac{\omega}{c} n$

then $k(\omega) = \frac{\omega}{c} n(\omega)$

then $\frac{1}{v_g} = \frac{dk}{d\omega} = \frac{1}{c}\left[n(\omega) + \omega \frac{dn}{d\omega}\right]$

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Or

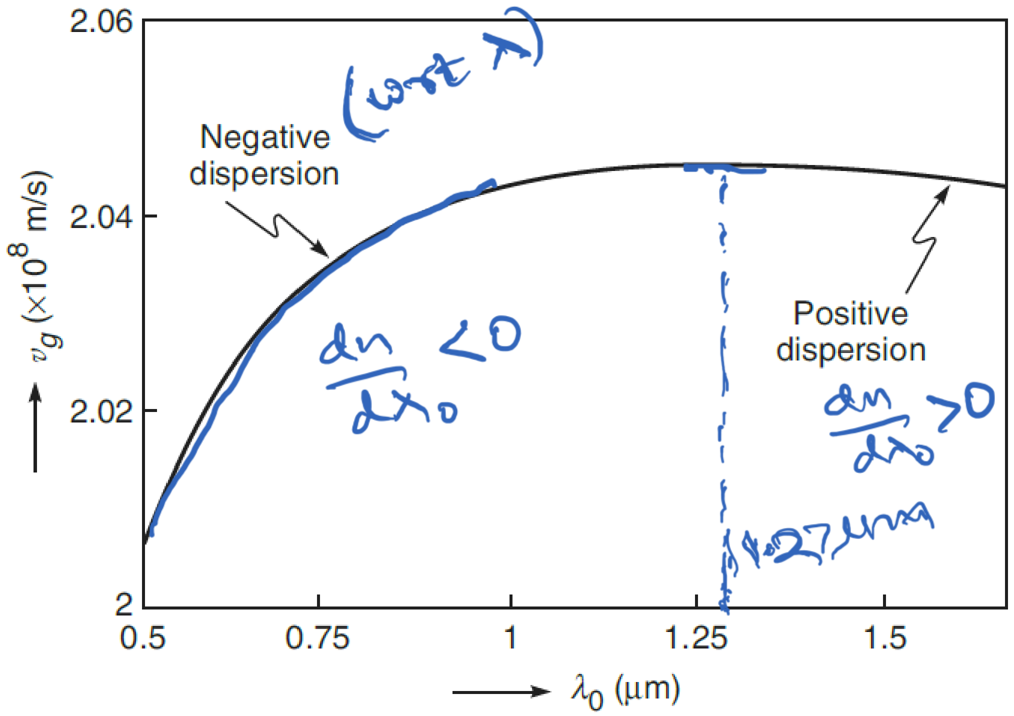
$= -\frac{\lambda_0^2}{2\pi c} \frac{dn}{d\lambda_0}$

then $\boxed{\frac{1}{v_g} = \frac{1}{c}\left[n(\lambda_0) - \lambda_0 \frac{dn}{d\lambda_0}\right]}$

$dn/d\lambda_0 \rightarrow$ dispersion

In a non dispersive medium $v_g = v_p$

GROUP VELOCITY AND GROUP INDEX



Variation of group velocity with wavelength for pure silica → Fibre core

Dispersive medium: the light with different wavelength travels with different group velocity

group index

$$n_g = c/v_g = \left[n(\lambda_0) - \lambda_0 \frac{dn}{d\lambda_0} \right]$$

$$v_g = c \left[n(\lambda_0) - \lambda_0 \frac{dn}{d\lambda_0} \right]^{-1}$$

when $\frac{dn}{d\lambda_0} > 0$ → positive dispersion then v_g decreases

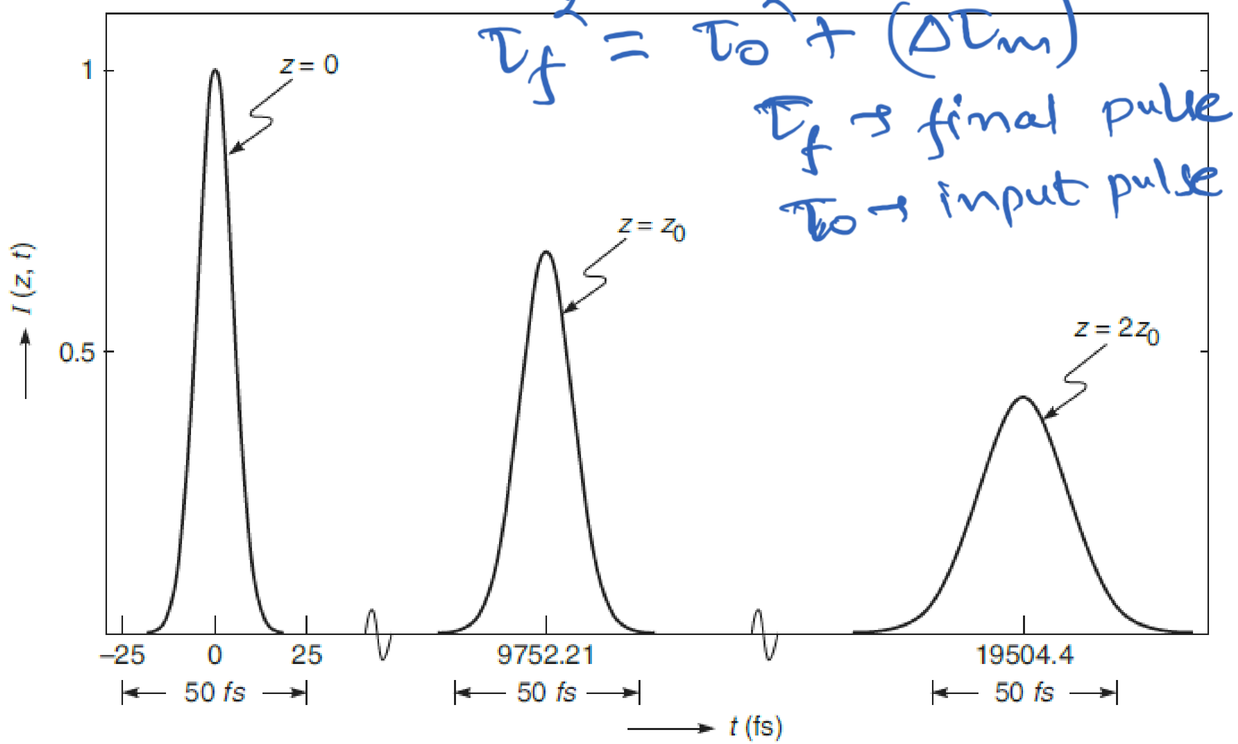
$dn/d\lambda_0 < 0$ → negative dispersion then v_g increases

For silica the group velocity becomes maximum at 1.27 μm

⇒ less dispersion losses

∴ Good for communication applications

PULSE BROADENING (DISPERSION)



$$\tau_f^2 = \tau_0^2 + (\Delta\tau_m)^2$$

$\tau_f \rightarrow$ final pulse

$\tau_0 \rightarrow$ input pulse

Time variation of pulse intensity at different values of z: Temporal broadening of the pulse in the positive dispersion regime of silica

If you consider a light pulse, that will have certain wavelength spread $(\Delta\lambda)$, since the each component of light travels with different group velocity, it will in general result in the broadening of the pulses.

The time taken by the pulse to traverse distance L of the dispersive medium is given by,

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Pulse broadening $\Delta\tau_m = \frac{d\tau}{d\lambda_0}\Delta\lambda_0$

$\Delta\tau_m \rightarrow$ material dispersion

GROUP VELOCITY DISPERSION (GVD)

$$\Delta\tau_m = -\frac{L\,\Delta\lambda_0}{\lambda_0 c}\left[\lambda_0^2 \frac{d^2n}{d\lambda_0^2}\right]$$

$$= L\, D_m\, \Delta\lambda_0$$

$$n = \frac{\lambda_0}{\lambda}$$

$$\frac{dn}{d\lambda} = -\frac{\lambda_0}{\lambda^2}$$

$$\frac{d^2n}{d\lambda^2} = \frac{2\lambda_0}{\lambda^3}$$

or $D_m = \dfrac{\Delta\tau_m}{L\,\Delta\lambda_0} = -\dfrac{\lambda_0}{c}\dfrac{d^2n}{d\lambda_0^2}$ → material dispersion coefficient

D_m has units → ps / km.nm

and $\dfrac{d^2n}{d\lambda^2}$ → Group velocity dispersion (GVD)

when $\dfrac{d^2n}{d\lambda_0^2} > 0 \Rightarrow D_m < 0$ → Anamalous dispersion

→ pulse compresses

→ lower wavelengths travel faster than higher wavelengths

when $\dfrac{d^2n}{d\lambda_0^2} < 0 \Rightarrow D_m > 0$ → Normal dispersion → pulse Broadens

→ lower wavelengths ($v \propto \lambda$) travel slower than longer wavelengths

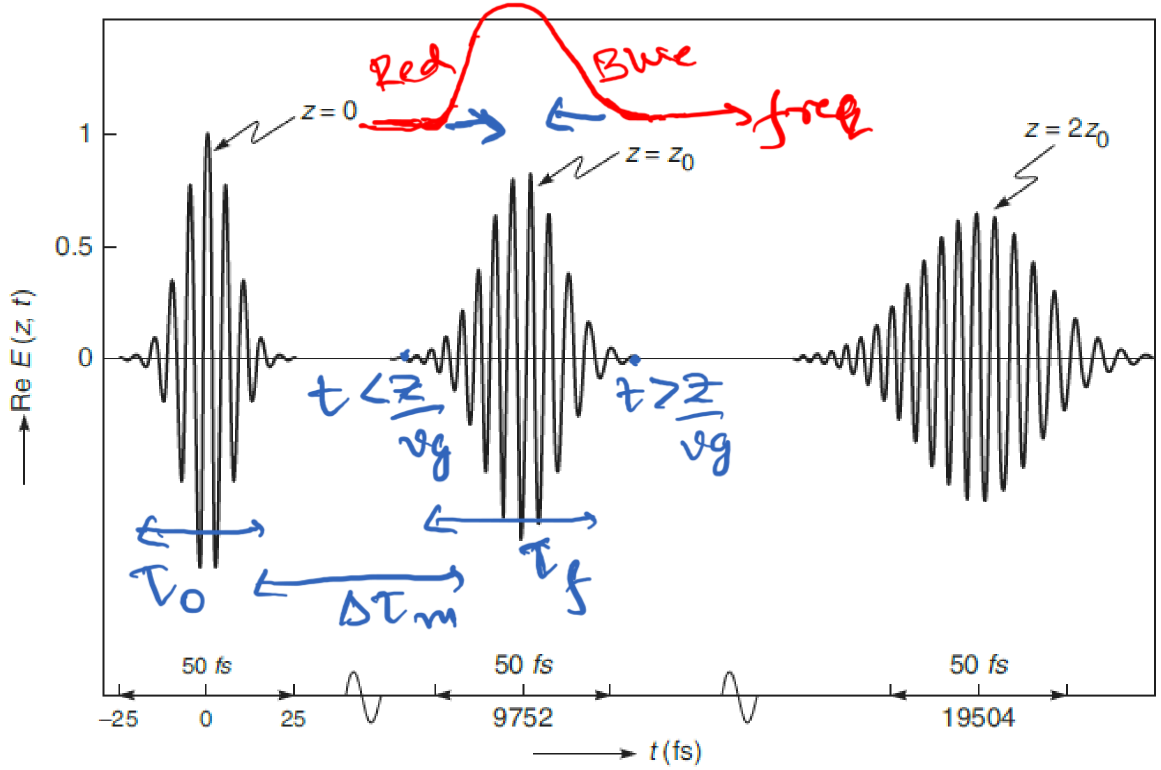
PULSE BROADENING (DISPERSION)

$v \propto \lambda$, $f \propto 1/\lambda$, $f \propto 1/t$,

The frequency chirp $\Delta\omega = \omega(t) - \omega_0$

$$= 2K\left(t - \frac{z}{v_g}\right)$$

$K \rightarrow$ phase term



Temporal broadening of the pulse in the positive dispersion regime of silica

$z \rightarrow$ distance

$v_g \rightarrow$ group velocity

For positive (normal) dispersion

$K < 0$

$\therefore$ If $t < \frac{z}{v_g} \rightarrow$ trailing edge

$\omega(t) > \omega_0 \Rightarrow$ lower frequency is blue shifted

If $t > \frac{z}{v_g} \rightarrow$ leading edge

$\omega(t) < \omega_0 \Rightarrow$ higher frequency is red shifted

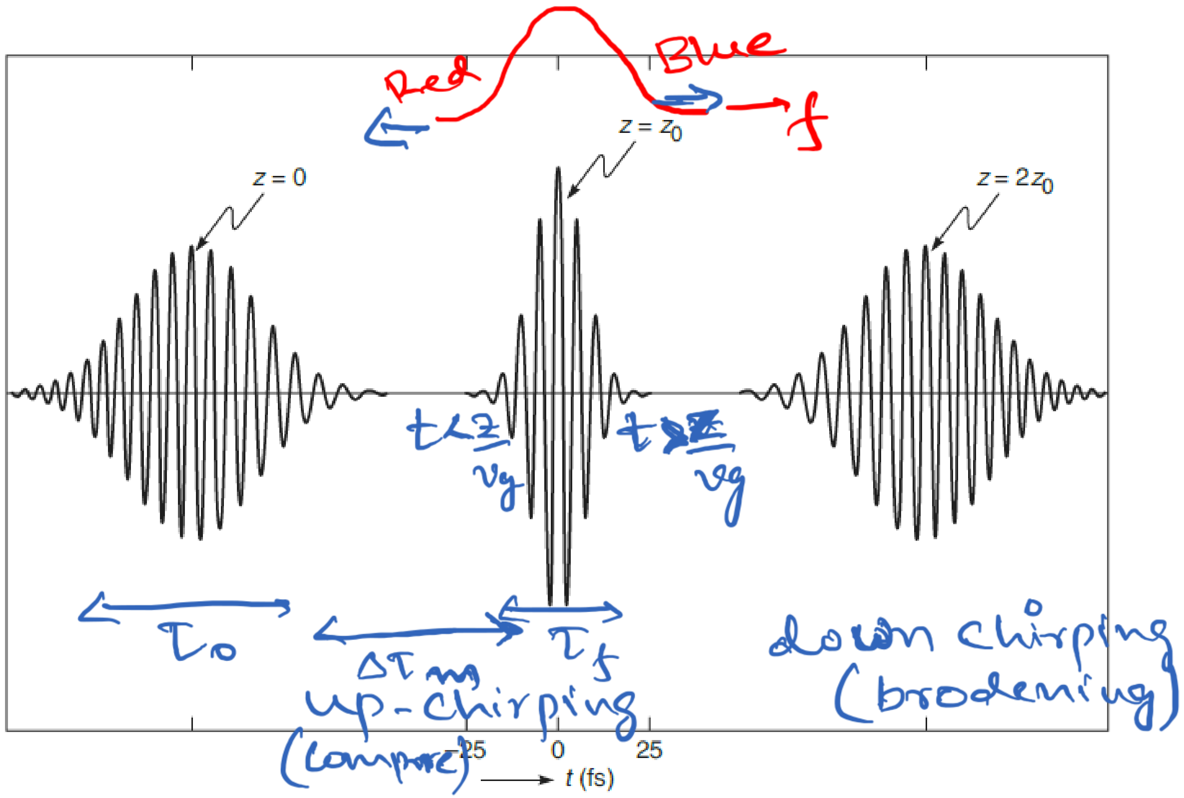
$\Rightarrow$ Frequency band decreases

$\Rightarrow$ Down chirping

Results in Broadening of pulse

$\tau_f > \tau_0$

PULSE BROADENING (DISPERSION)



Temporal compression in the negative dispersion regime and broadening in the positive dispersion regime of silica

For negative (anamalous) dispersion

$K > 0$

$\therefore$ if $t < \frac{z}{v_g}$ $\Rightarrow$ trailing edge

$\omega(t) < \omega_0$ $\Rightarrow$ lower frequency is red shifted

if $t > z/v_g$ $\Rightarrow$ leading edge

$\omega(t) > \omega_0$ $\Rightarrow$ higher frequency is blue shifted.

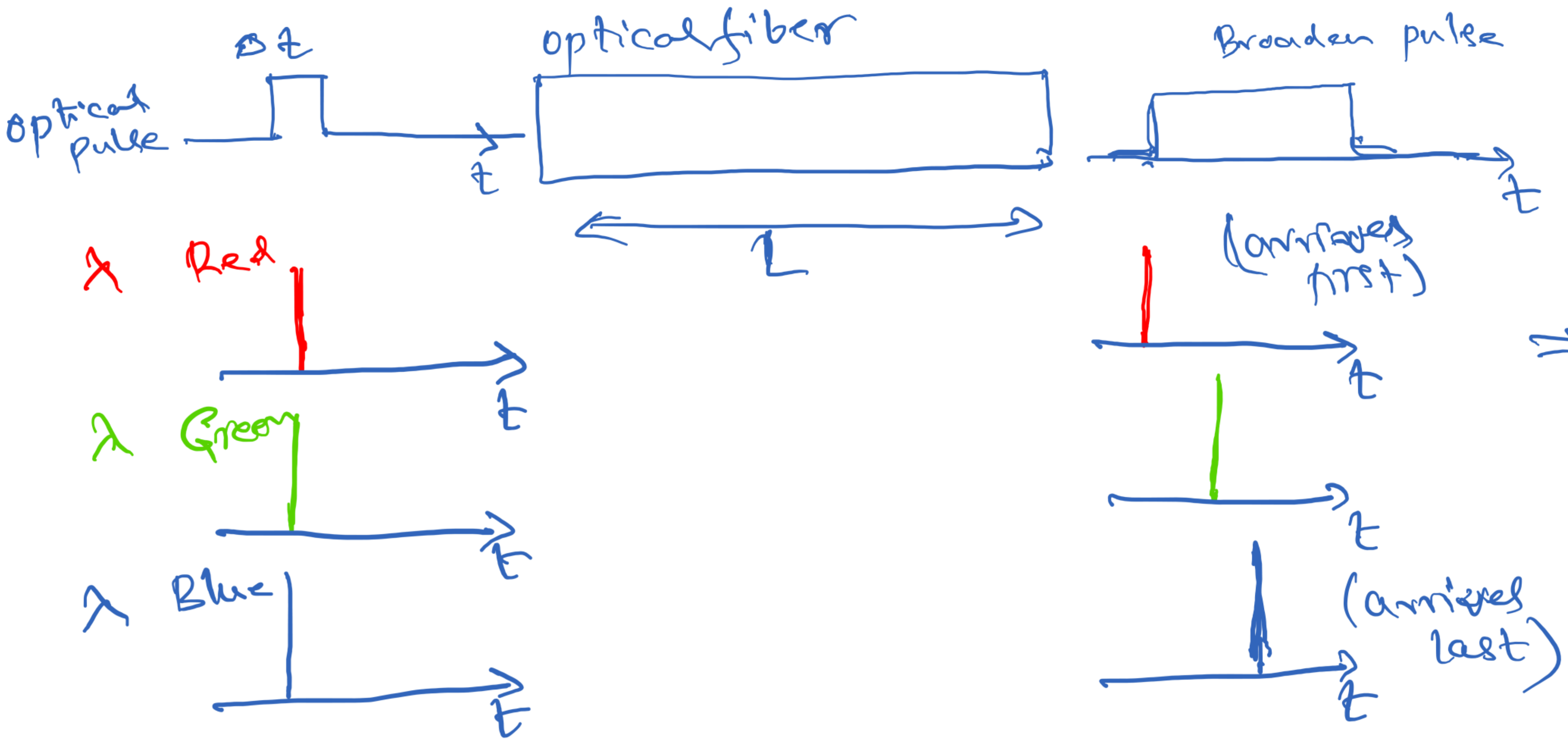
$\Rightarrow$ The frequency band increases

$\rightarrow$ up chirping

Results in compression of pulse

i.e $T_f < T_0$

PROPAGATION OF THE PULSE IN OPTICAL FIBER

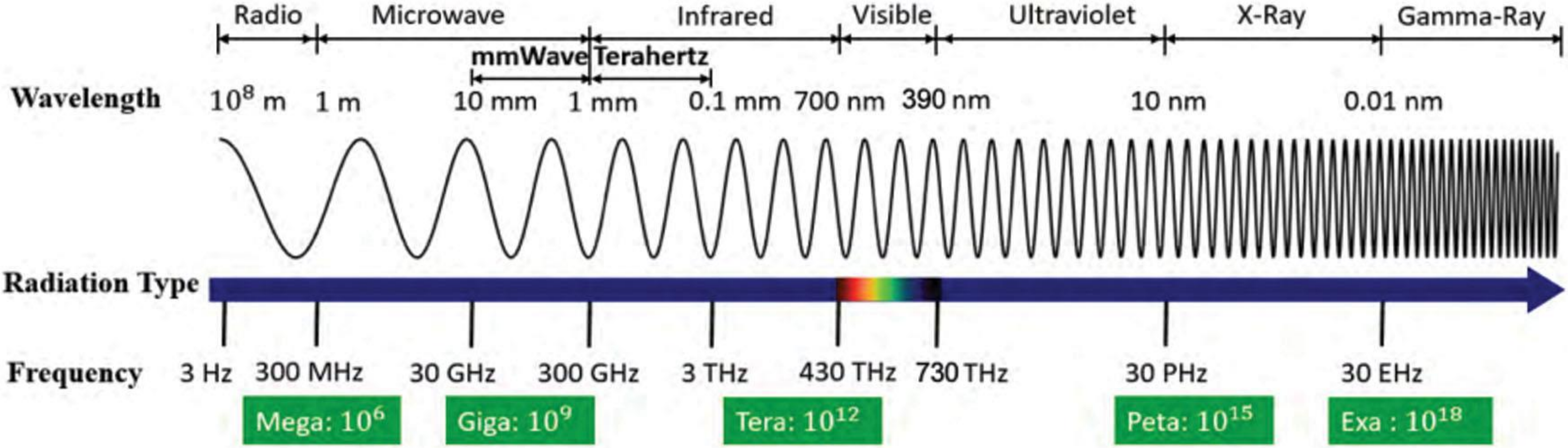


$n \propto 1/\lambda$

⇒ longer wavelength will have higher velocity and arrives first

→ Consequence is we need to increase the time seperation b/w the input pulses (causes reduced data rate) to avoid the overlapping of pulses due to dispersion.

ELECTROMAGNETIC WAVES



Spectral Region	Approximate Frequency Range
Gamma rays	$>10^{20}$ Hz
X-rays	$10^{17}-10^{20}$ Hz
Ultraviolet	$10^{15}-10^{17}$ Hz
Visible	$(3.5-7.5)\times10^{14}$ Hz
Infrared	$10^{12}-10^{14}$ Hz
Microwaves	$10^{9}-10^{12}$ Hz
Radiofrequency	$<10^{9}$ Hz

→ Light is a composite of electric and magnetic oscillations propagating in space and time

→ Electric field, magnetic field are perpendicular to each other and lie on the plane transverse to the direction of propagation.

THE LIGHT WAVES: Maxwell's Equations

→ Unification theory of electricity and magnetism by James C. Maxwell in 1865.

→ Describes the vector form of the waves (Light waves)

Maxwell's equations in an isotropic/homogenous medium

$$\vec{\nabla} \cdot \vec{E} = \rho/\epsilon_0 \quad \text{--- (1)}$$

$$\vec{\nabla} \cdot \vec{B} = 0 \quad \text{--- (2)}$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad \text{--- (3)}$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \left[\vec{J} + \epsilon_0 \frac{\partial \vec{E}}{\partial t} \right] \quad \text{--- (4)}$$

ρ → charge density

J → current density

ϵ_0 → permittivity in free space

μ_0 → permeability in free space.

$\epsilon_0 = 8.85 \times 10^{-12}$ F/m

$\mu_0 = 4\pi \times 10^{-7}$ Ns^2/C^2

THE WAVE EQUATION

→ One can derive wave equation for the EM waves using Maxwell's equations.

Taking curl on $\vec{\nabla} \times \vec{E}$ gives for homogenous medium ($\rho = J = 0$)

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = \vec{\nabla} \times \left(-\frac{\partial \vec{B}}{\partial t}\right)$$

$$\vec{\nabla}\underbrace{(\vec{\nabla} \cdot \vec{E})}_{0} - \nabla^2 \vec{E} = -\frac{\partial (\vec{\nabla} \times \vec{B})}{\partial t}$$

$$-\nabla^2 \vec{E} = -\mu_0 \epsilon_0 \frac{\partial^2 E}{\partial t^2}$$

→ linear
→ homogeneous
→ second order diff. equation.

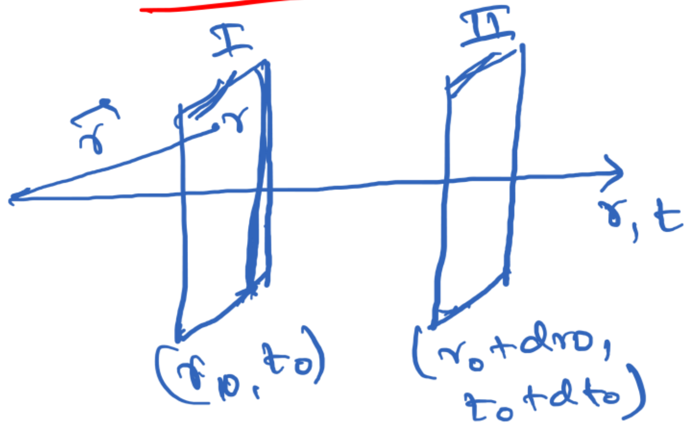
$$\boxed{\nabla^2 \vec{E} - \mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} = 0}$$ → wave equation of EM wave

$$\boxed{\nabla^2 = \frac{\partial}{\partial x^2} + \frac{\partial}{\partial y^2} + \frac{\partial}{\partial z^2}}$$

$\parallel^{ly}$ $$\boxed{\nabla^2 \vec{B} - \mu_0 \epsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2} = 0}$$

$\frac{1}{\sqrt{\mu_0 \epsilon_0}} = c$ → velocity of light in vacuum.

Plane waves



→ General solution of the wave equation

$$\vec{E}(\vec{r},t) = \vec{E}_0(\vec{r},t)\, e^{-i\phi(\vec{r},t)}$$

where, $\phi(\vec{r},t) = (\vec{k}\cdot\vec{r} - \omega t)$ → Phase of the wave

For a plane wave, the phase (ϕ) of the wavefront at different space-time should be equal (constant)

i.e. $\phi_0 = k r_0 - \omega t_0$ at plane I

& $\phi_0 = k(r_0 + dr_0) - \omega(t_0 + dt_0)$ at plane II should be equal

$$\Rightarrow \quad k r_0 - \omega t_0 = k(r_0 + dr_0) - \omega(t_0 + dt_0)$$

Transversality of $\vec{E}$, $\vec{H}$ & $\vec{k}$ using $\vec{\nabla}\cdot\vec{E} = 0$

$$\boxed{\vec{\nabla} = \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}}$$

$$\vec{\nabla}\cdot\vec{E} = i\,(k_x E_x + k_y E_y + k_z E_z)\, e^{i(k\cdot r - \omega t)} = 0$$

$\Rightarrow \quad \vec{k}\cdot\vec{E} = 0$ similarly $\vec{k}\cdot\vec{H} = 0$

$\Rightarrow \quad \vec{E}, \vec{H}$ and $\vec{k}$ are $\perp^r$ to each other.

HOMOGENEOUS AND INHOMOGENEOUS WAVES

Solns to the E.M wave equations

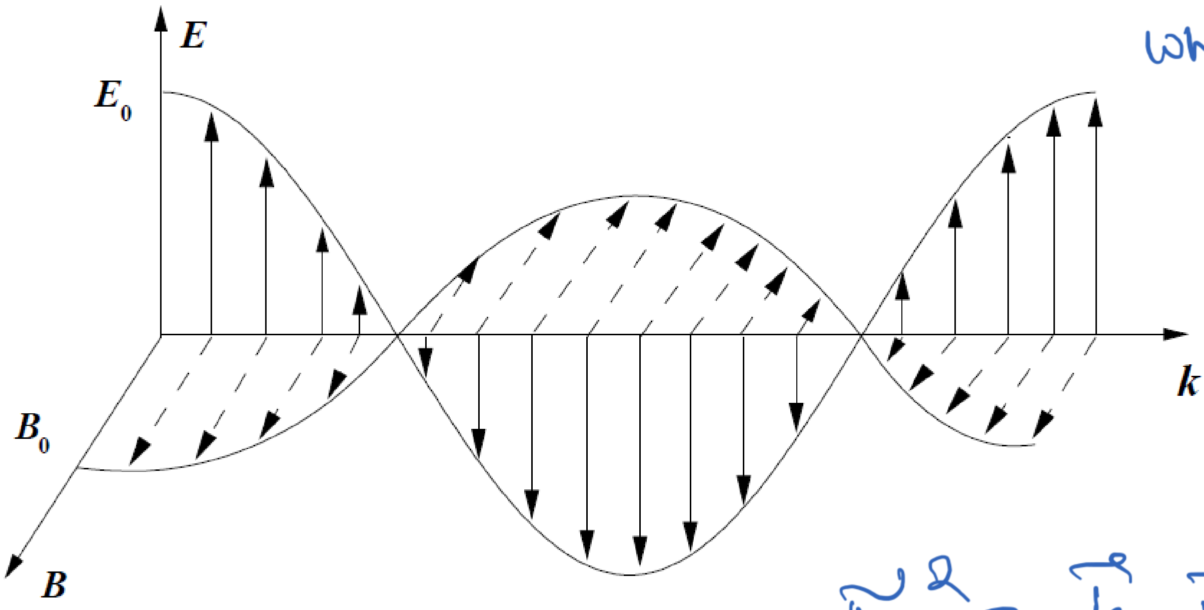
$$\vec{E}(\vec{r},t) = \vec{E}_0 \, e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$

$$\vec{B}(\vec{r},t) = \vec{B}_0 \, e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$

plane wave soln.

$$\boxed{\vec{E} \perp^r \vec{B} \perp^r \vec{k}}$$

Transverse wave



$\vec{k}$ is the propagation vector

$$\vec{\tilde{k}} = \vec{k} + i\vec{\alpha} \quad \text{(Complex form)}$$

Where, $\vec{\alpha}$ is the attenuation vector (depends on the medium) (absorption)

$$\tilde{k} = \frac{\omega}{c}\tilde{n}$$

$\tilde{n} \rightarrow$ complex refractive index

$$\tilde{n} = n + iK ,$$

$K \rightarrow$ extinction coefficient (absorption)

$$\tilde{k}^2 = \vec{\tilde{k}}\cdot\vec{\tilde{k}} = \frac{\omega^2}{c^2}\tilde{n}^2$$

HOMOGENEOUS AND INHOMOGENEOUS WAVES

⇒ Two relations $k^2 - \alpha^2 = (n^2 - K^2)\,\omega^2/c^2$

and $\vec{k}\cdot\vec{\alpha} = nK\,\omega^2/c^2$

For a purely homogeneous (transparent) medium,

$\vec{k}\cdot\vec{\alpha} = 0$ → There are two possibilities that satisfy this relation.

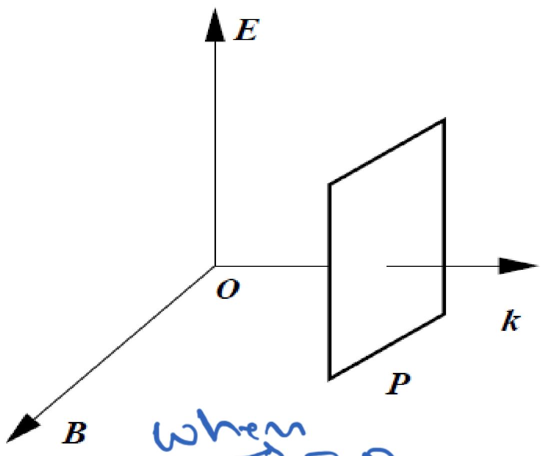
<u>Case I</u> :- $\vec{\alpha} = 0$ (no-absorption)

then, $k = n\omega/c$,

and the plane wave soln, takes the form

$\vec{E} = \vec{E}_0\, e^{i(\vec{k}\cdot\vec{r} - \omega t)}$ → plane wave with velocity $v = \omega/k = c/n$

& $\vec{B} = \vec{B}_0\, e^{i(\vec{k}\cdot\vec{r} - \omega t)}$



when $\vec{\alpha} = 0$

→ homogeneous plane wave with constant phase and constant amplitude.

HOMOGENEOUS AND INHOMOGENEOUS WAVES

Case - II :- $\vec{k} \cdot \vec{a} = 0$, when $\vec{k} \perp^r \vec{a}$, but $\vec{a} \neq 0$,

Then the plane wave solution takes the form

$$\vec{E}(\vec{r}, t) = \vec{E}_0 \, e^{-\vec{a} \cdot \vec{r}} \cdot e^{i(\vec{k} \cdot \vec{r} - \omega t)}$$

→ Now the wave propagates in direction $\vec{k}$ but with diminished velocity as compared to the velocity of homogeneous wave with $\vec{a} = 0$.

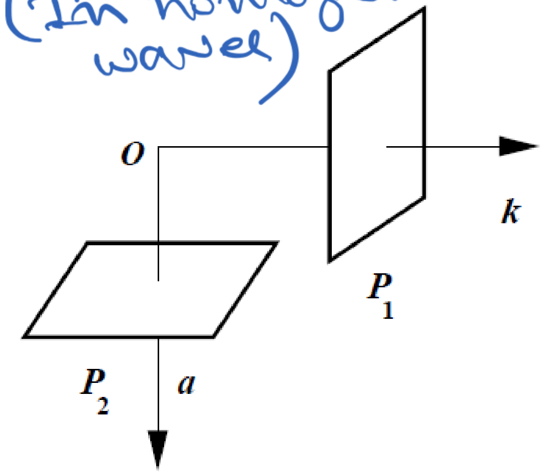
→ Here the constant amplitude and phase are never satisfied

→ Inhomogeneous wave

the propagates with decreased amplitude in the direction of $\vec{a}$.

→ Planes with constant amplitude $\perp^r \vec{a}$ & planes with constant phase $\perp^r \vec{k}$

Ex:- Evanescent waves. (Inhomogeneous wave)

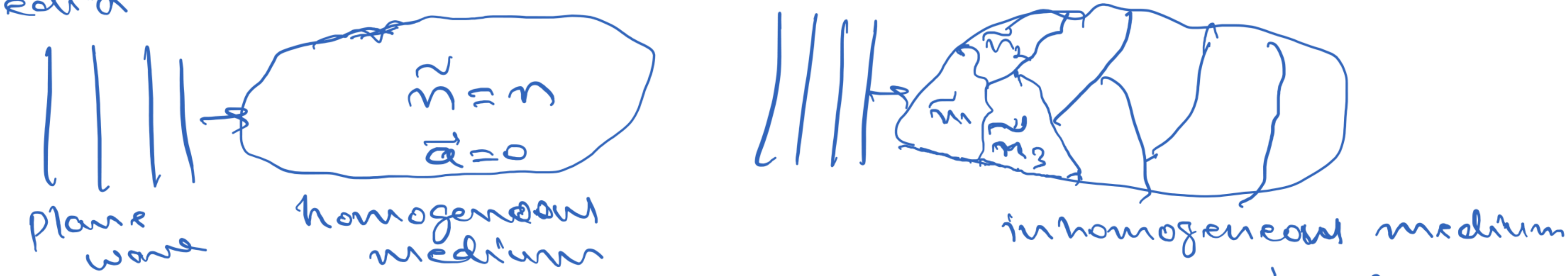


BOUNDARY CONDITIONS

Continuity of Normal and Tangential Components of EM waves at the interface

→ So far we have been considering the wave propagation in a source free infinite homogeneous medium.

→ But in practice we will encounter wave propagation in a medium of finite extent and it is important to look at the wave equation at interface between the media



→ Maxwell's equation satisfies the Gauss and Stoke's theorems everywhere in the two media & in the region of the interface between them → the restrictions imposed by these laws on E & B fields at two sides of interface → continuity / boundary conditions.

BOUNDARY CONDITIONS

Continuity of Normal Components of EM waves at the interface

→ ϵ_1 and ϵ_2 are the permittivity of two media

→ consider infinitisimally small cylinder of height (h) at the interface of ϵ_1 & ϵ_2.

→ D_1 & D_2 are the displacement fields due to ϵ_1 & ϵ_2.

Applying Gauss theorem on the cylinder

$$\boxed{\vec{D} = \epsilon \vec{E}}$$

$$\oiint_S \vec{D} \cdot d\vec{A} = \iiint_V \vec{\nabla} \cdot \vec{D} \, dv.$$

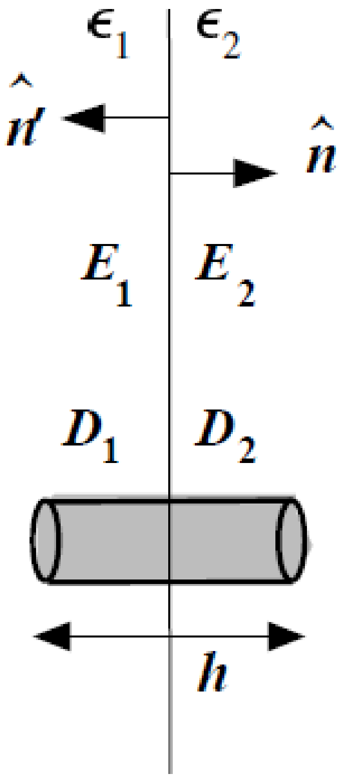
The RHS vanishes as $h \to 0$, $v \to 0$

and LHS will be vanishingly small

Therefore, $\epsilon_1 \vec{E}_1 \cdot \hat{n}' + \epsilon_2 \vec{E}_2 \cdot \hat{n} = 0$

$\hat{n}' = -\hat{n}$ $\Rightarrow$ $\boxed{\epsilon_1 E_1^{\perp} \cdot \hat{n} = \epsilon_2 E_2^{\perp} \cdot \hat{n}}$

→ continuity of the normal components of E-fields

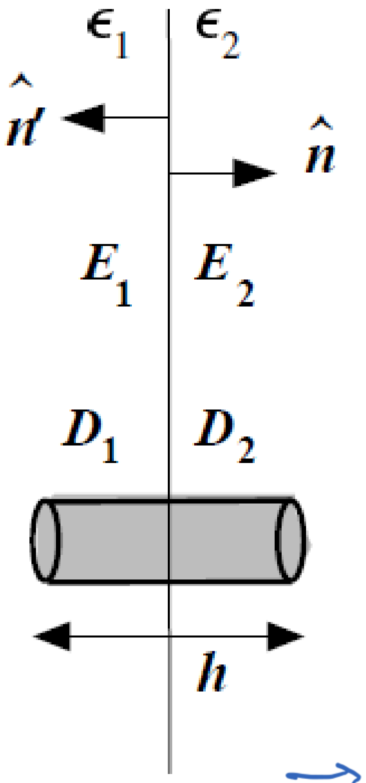


(a)

Plane boundary between two homogeneous media

BOUNDARY CONDITIONS

Continuity of Normal Components of EM waves at the interface



(a)

Plane boundary between two homogeneous media

Similarly, $\vec{B}_1^{\perp} \cdot \hat{n} = \vec{B}_2^{\perp} \cdot \hat{n}$

→ In the case of linear media

$$\epsilon_1 \vec{E}_1^{\perp} \cdot \hat{n} - \epsilon_2 \vec{E}_2^{\perp} \cdot \hat{n} = \rho_f$$

ρ_f → free charge density

and $\vec{B}_1^{\perp} \cdot \hat{n} = \vec{B}_2^{\perp} \cdot \hat{n}$

→ the normal component of the electric field at the interface b/w the linear media is discontinuous by an amount ρ_f

→ the normal component of B-field remains continuous.

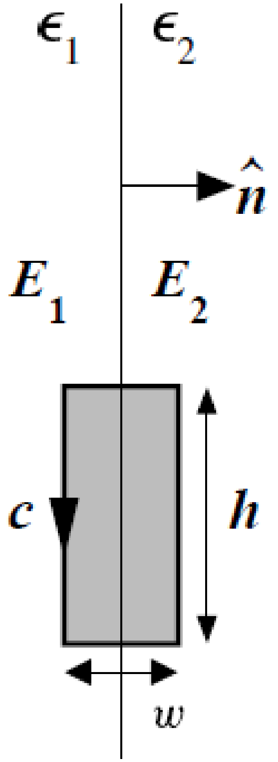
BOUNDARY CONDITIONS

Continuity of Tangential Components of EM waves at the interface

Here, we apply the stoke's theorem

$$\oint_C \vec{E}\cdot d\vec{l} = \iint_S \vec{\nabla}\times\vec{E}\cdot d\vec{A} \qquad \left[\vec{\nabla}\times\vec{E} = -\frac{\partial B}{\partial t}\right]$$

$$= -\frac{\partial}{\partial t}\iint_\Sigma \vec{B}\cdot d\vec{A}$$



(b)

Plane boundary between two homogeneous media

C → closed path that encloses the boundary between two media

Σ → surface bounded by closed path "C"

w → width of the cylinder

h → height of the cylinder (Gaussian)

As $\omega \to 0$, RHS → 0 (surface integral)

leading to the continuety of the tangential components of E & B fields across the boundary.

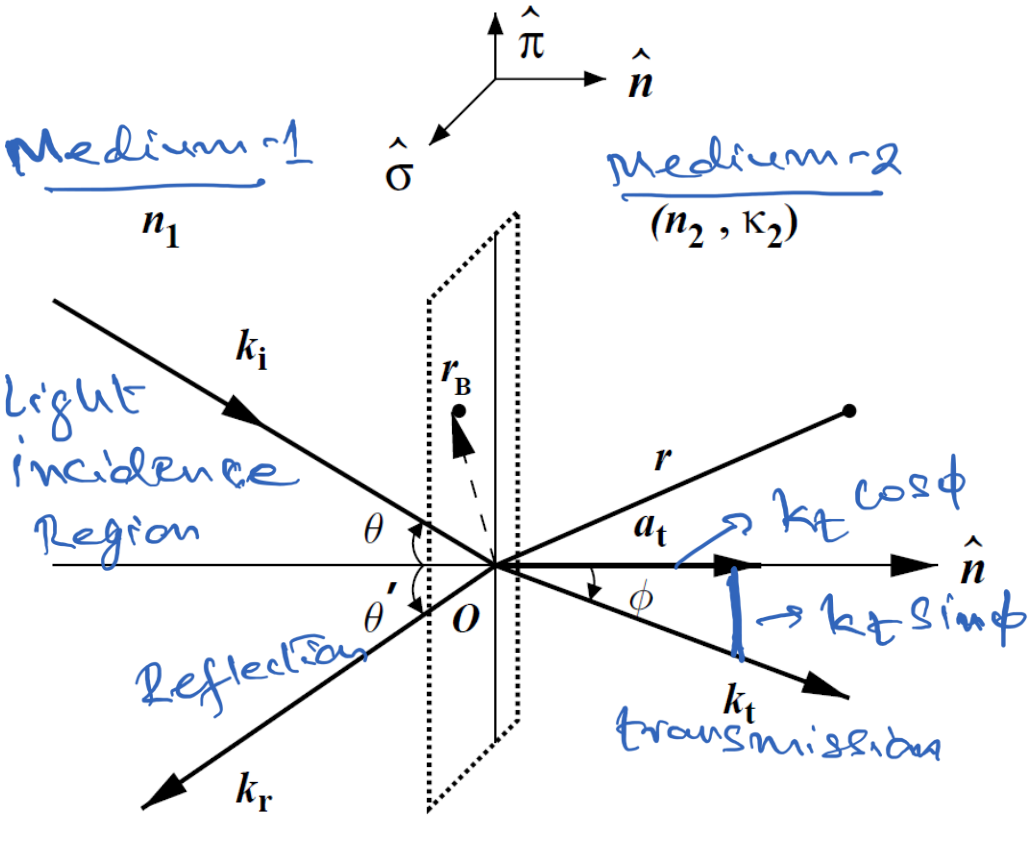
$$\vec{E}_1^{\parallel}\times\hat{n} = \vec{E}_2^{\parallel}\times\hat{n}$$

III^ly

$$\vec{H}_1^{\parallel}\times\hat{n} = \vec{H}_2^{\parallel}\times\hat{n}$$

} continuety of the tangential components

REFLECTION AND TRANSMISSION AT THE BOUNDARY



Reflection and transmission of a wave at a plane boundary.

$\vec{r}_B \to$ position vector of a point on the plane of the boundary

$\hat{n} \to$ unit vector normal to the plane of the boundary

In the figure, refractive index 'n_1' represents the medium of incident light. It is perfectly transparent ($\kappa_1 = 0$) perfectly non-absorbing ($\vec{a}_1 = 0$)

Medium-2, has refractive index

$\tilde{n}_2 = n_2 + i\kappa_2$ $\quad \kappa \to$ extinction coefficient

& $\tilde{k} = k_t + i a_t$ $\quad \vec{a} \to$ attenuation coefficient

Now from boundary conditions of EM waves at the interface,

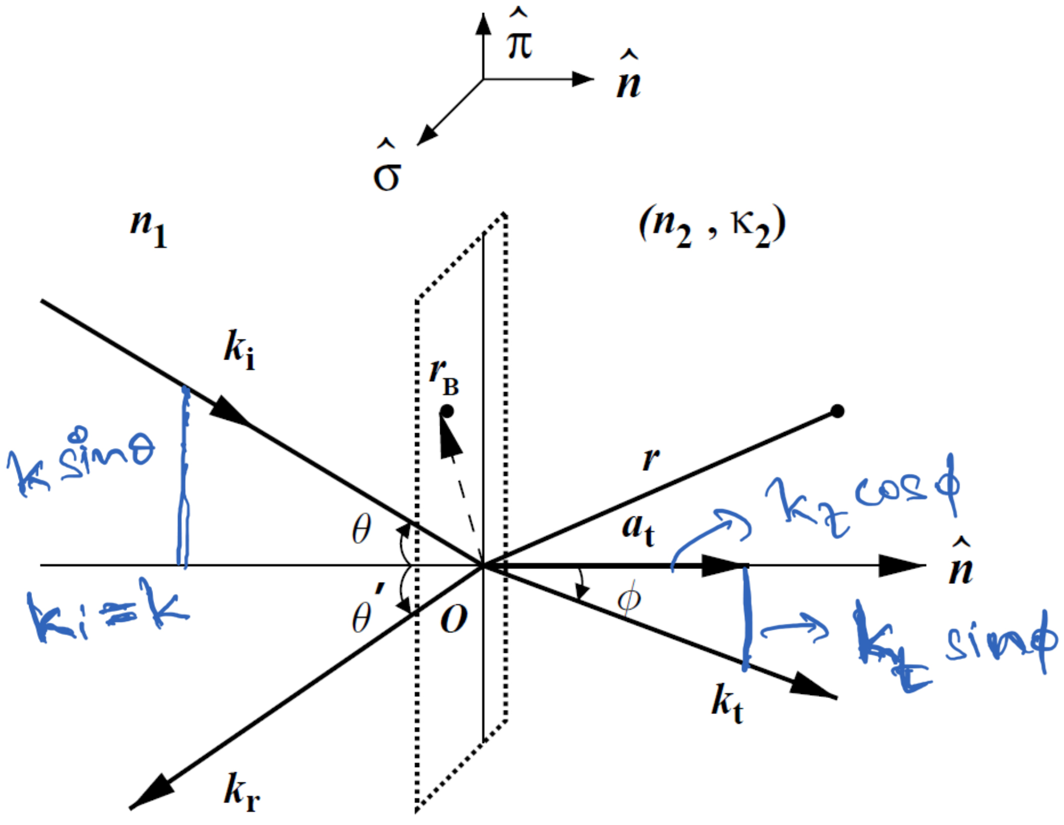
$$\vec{k}_i \cdot \vec{r}_B = \vec{k}_r \cdot \vec{r}_B = \vec{k}_t \cdot \vec{r}_B$$

(normal components)

$$\vec{k}_i \times \hat{n} = \vec{k}_r \times \vec{n} = \vec{k}_t \times \vec{n}$$

(tangential components)

REFLECTION AND TRANSMISSION AT THE BOUNDARY



Reflection and transmission of a wave at a plane boundary.

We have $\theta = \theta'$ → law of reflection

and $k_t \sin\phi = k \sin\theta$ → Fermat's principle

θ → angle of incidence

θ' → angle of reflection

ϕ → angle of refraction

We recall, $\tilde{k} = \frac{\omega}{c}\tilde{n}$

where $\tilde{k} = \vec{k} + i\vec{a}$

and $\tilde{n} = n + i\kappa$

$\tilde{k}^2 = \frac{\omega^2}{c^2}\tilde{n}^2$

At the interface, this equ. becomes

$$(k_t\cos\phi + i a_t)^2 + (k_t\sin\phi)^2 = \frac{\omega^2}{c^2}(n_2 + i\kappa_2)^2$$

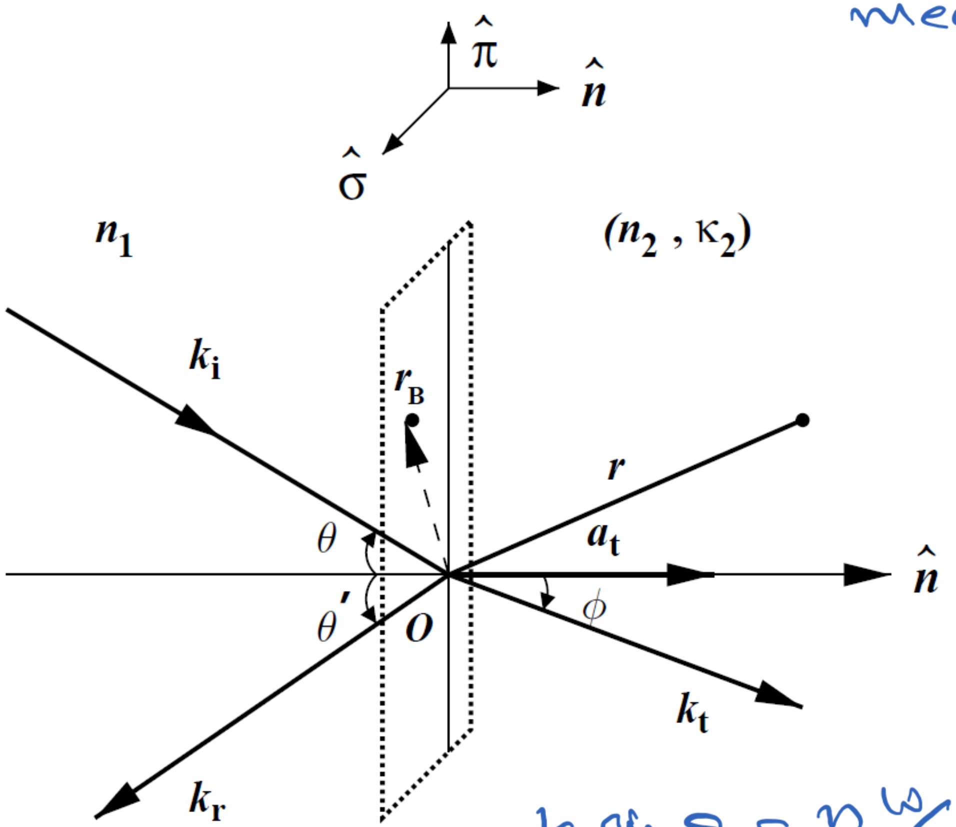
$$\Rightarrow (k_t\cos\phi + i a_t)^2 + (k\sin\theta)^2 = \frac{\omega^2}{c^2}(n_2 + i\kappa_2)^2$$

EXTERNAL REFLECTIONS

→ Here we consider the case in which the light crosses the interface from a rarer medium to optically denser medium (i.e $n_1 < n_2$)

→ We observe reflection and refraction phenomena.

→ Reflections in this case is reffered to as "external reflections".



Reflection and transmission of a wave at a plane boundary.

$$\boxed{k \sin\theta = n \frac{\omega}{c} \sin\theta}$$

If the second medium is also perfectly transparent and non-absorbing (i.e, $K=0$ & $a_t=0$)

Then, $k_t \cos\phi = \frac{\omega}{c}(n_2^2 - n_1^2 \sin^2\theta)^{1/2}$

if n_2 medium is homogeneous, then $k_t = n_2 \omega/c$

After substituting and simplifying we get,

under the condition of external reflections (i.e $n_1 < n_2$) → $\boxed{n_2 \sin\phi = n_1 \sin\theta}$ → Snell's law

Recap

→ Homogeneous and inhomogeneous waves

$(\vec{\alpha} = 0)$ $(\vec{\alpha} \neq 0)$ $\vec{\alpha}$ → attenuation vector

No absorption

plane waves with constant amplitude and phase $\parallel^{el}$ to $\vec{k}$

absorption

plane wave with constant amplitude $\parallel^{el}$ $\vec{k}$

constant phase $\parallel^{el}$ $\vec{\alpha}$

amplitude $\propto e^{-\vec{\alpha}\cdot\vec{r}}$.

(inside the medium)

→ Boundary conditions (continuity of $\vec{E}$ & $\vec{B}$ fields) at the interface.

for $\rho = 0$ ←

$$\left.\begin{aligned} \epsilon_1 \vec{E}_1^{\perp}\cdot\hat{n} &= \epsilon_2 \vec{E}_2^{\perp}\cdot\hat{n} \\ \vec{B}_1^{\perp}\cdot\hat{n} &= \vec{B}_2^{\perp}\cdot\hat{n} \end{aligned}\right\} \text{normal components}$$

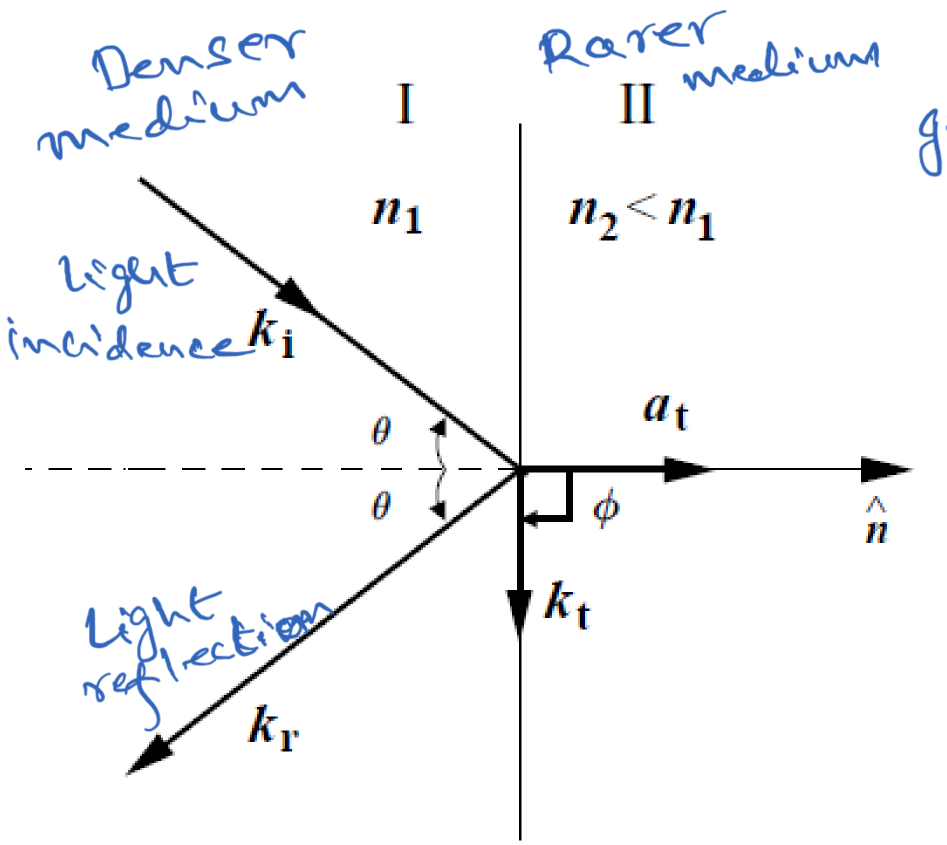
$$\left.\begin{aligned} \vec{E}_1^{\parallel}\times\hat{n} &= \vec{E}_2^{\parallel}\times\hat{n} \\ H_1^{\parallel}\times\hat{n} &= H_2^{\parallel}\times\hat{n} \end{aligned}\right\} \text{Tangential components.}$$

$\vec{H} = \vec{B}/\mu$

→ External reflections

$$\left(k_t \cos\phi + i\,\alpha_t\right)^2 + \left(k\sin\theta\right)^2 = \frac{\omega^2}{c^2}\left(n_2 + i\,\kappa_2\right)^2$$

INTERNAL REFLECTIONS



→ Here we consider the condition, where $n_1 > n_2$ (Light propagation from denser to rarer medium)

then we have the expression at the interface given by,

$$k_t \cos\phi + i a_t = \frac{\omega}{c}\left(n_2^2 - n_1^2 \sin^2\theta\right)^{1/2} \quad (1)$$

Here, RHS can not remain real for all angles of incidence.

→ Beyond certain angle called 'critical angle' (θ_c), the RHS becomes purely imaginary, i.e. at $\boxed{n_2 = n_1 \sin\theta_c}$

critical angle of incidence → $\boxed{\theta_c = \sin^{-1}\left(\frac{n_2}{n_1}\right)}$

TOTAL INTERNAL REFLECTION AND EVANESCENT WAVES

Case - I, when $\theta < \theta_c$, then we have the attenuation constant $\vec{a}_t = 0$ and the wave remains homogeneous $\longrightarrow$ External reflections

$\rightarrow$ But here since $n_1 > n_2$, the angle of refraction will be greater than angle of incidence (i.e light is refracted away from the normal)

Case - II when $\theta = \theta_c$ $\Rightarrow$ RHS $= 0$

$$\Rightarrow \quad k_t \cos\phi + i a_t = 0$$

since in homogenous condition $a_t = 0$,

$\Rightarrow k_t \cos\phi = 0 \Rightarrow \boxed{\phi = 90^0}$ $\rightarrow$ angle of refraction at the critical angle (θ_c)

$\Rightarrow$ the wave propagation in the second medium takes place at the interface only $\longrightarrow$ Evanescent waves

TOTAL INTERNAL REFLECTION AND EVANESCENT WAVES

Case III :- when $\theta > \theta_c$, the light is totally reflected back into the medium (n_1)

→ Total internal reflection (TIR)

→ Evanescent wave continues to propagate at the interface.

→ For $\theta > \theta_c$, the RHS in eqn (1) becomes purely imaginary. and the transmitted wave (k_t) continues to propagate along the interface with propagation vector of magnitude

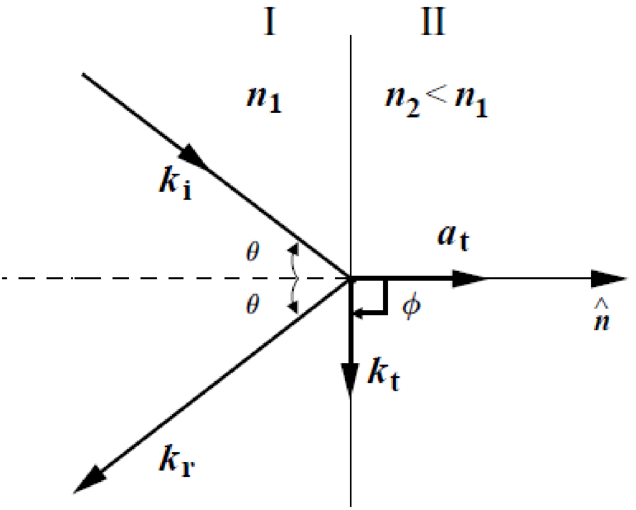
$$k_t = n_1 \frac{\omega}{c} \sin\theta$$

from $k_t \sin\phi = n \frac{\omega}{c} \sin\theta$ (Fermat's principle) $\sin(90^\circ) = 1$

now with an attenuation vector $\vec{a}$ of magnitude

$$a_t = \omega/c \left(n_1^2 \sin^2\theta - n_2^2\right)^{1/2}$$

→ directed normal to the plane of the boundary.



TOTAL INTERNAL REFLECTION AND EVANESCENT WAVES

The equation for transmitted wave takes the form

$$\vec{E}_{tr} = \vec{E}_t \, e^{i[(k_t + i a_t)\cdot \vec{r} - \omega t]}$$

$$\vec{E}_{tr} = \vec{E}_t \, e^{-\frac{\omega}{c}(n_1^2 \sin^2\theta - n_2^2)^{1/2}\,\hat{y}} \cdot e^{i\left(\frac{n_1 \omega}{c} x \sin\theta - \omega t\right)}$$

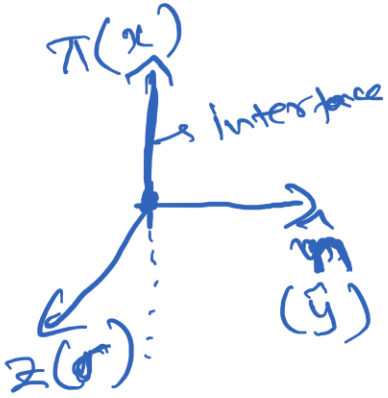
Amplitude of the evanescent wave decreases exponentially in the second medium (n_2) with $\hat{y}$ (along the direction of $\hat{n}$)

Transmitted evanescent wave propagates in the x-direction

The amplitude decreases to $1/e$ of its value at the interface at a distance away from the interface

$\lambda_0 \rightarrow$ incident light wave length

$k = 2\pi/\lambda_0$

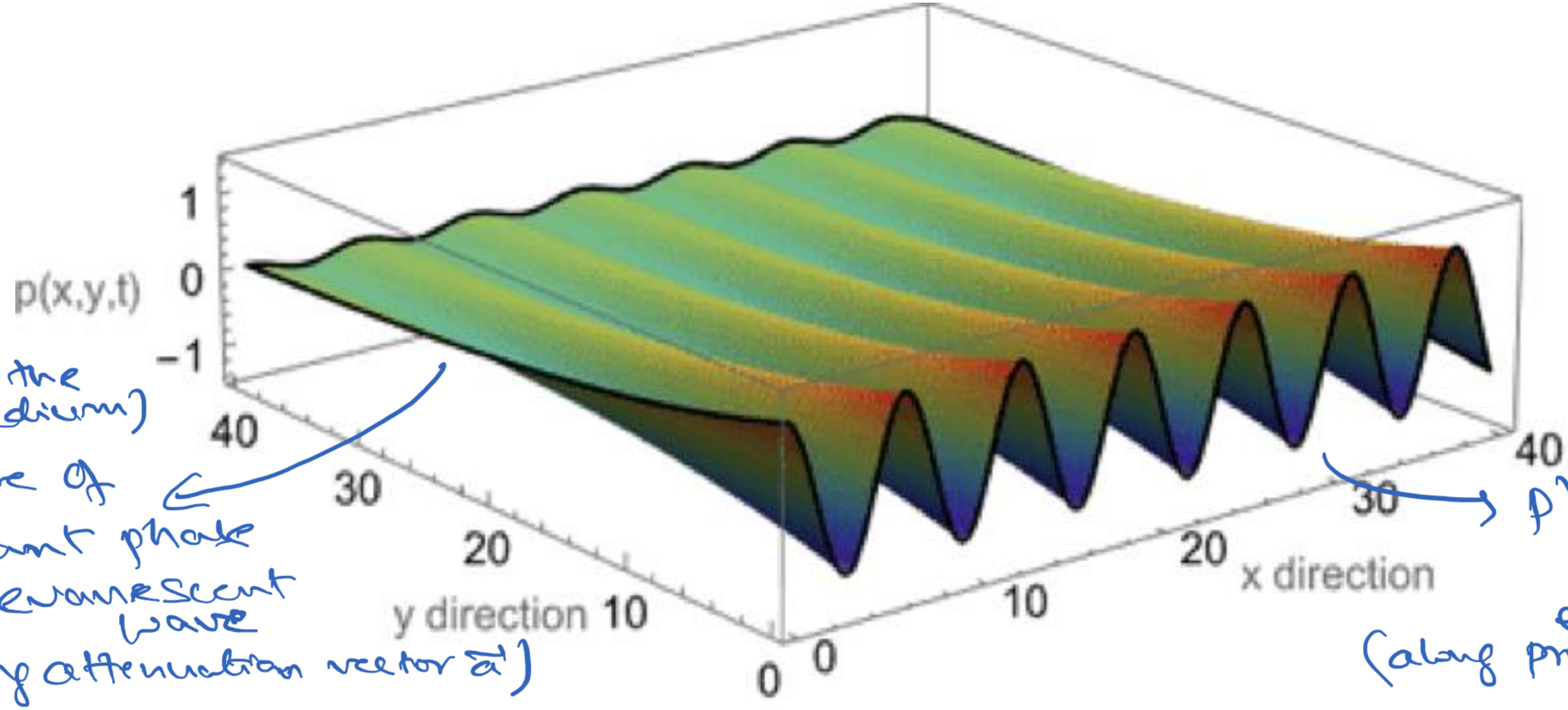


Penetration depth $\rightarrow$ $\delta = \dfrac{1}{a_t} = \dfrac{\lambda_0}{2\pi\,(n_1^2 \sin^2\theta - n_2^2)^{1/2}}$

$\rightarrow$ The beam attenuation increases with increasing angle of incidence (θ) beyond its critical angle (θ_c)

$\rightarrow$ penetration depth (δ) decreases with increasing θ and $\delta \ll \lambda_0$

TOTAL INTERNAL REFLECTION AND EVANESCENT WAVES



(inside the medium)
plane of constant phase of evanescent wave (along attenuation vector $\vec{\alpha}$)

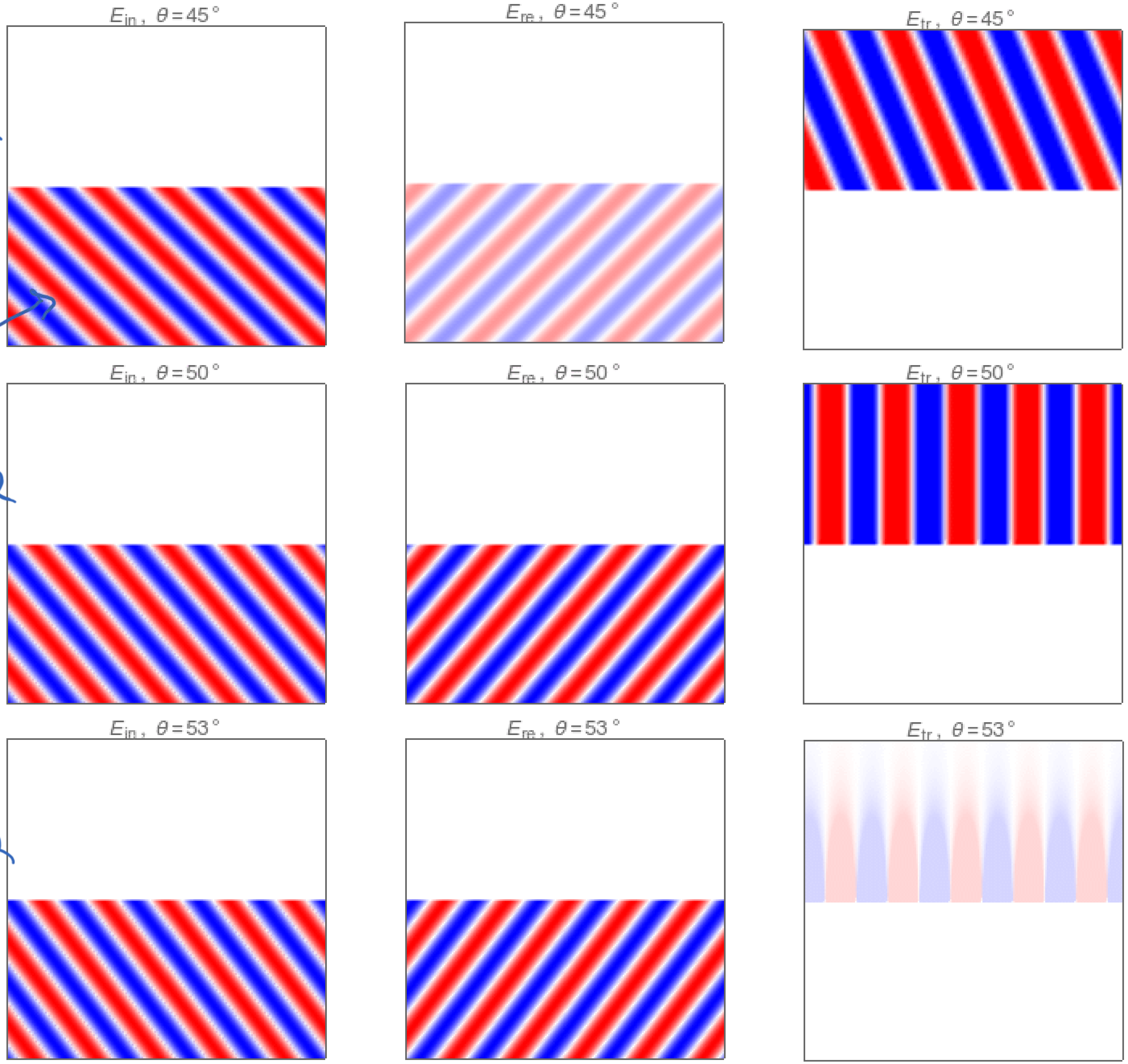
plane of constant amplitude of evanescent wave (along propagation direction)

→ For evanescent waves, the wave fronts of constant phase are normal to the plane of the interface (along y) and the surfaces of constant amplitude are parallel to the plane of the interface (along x, propagation direction)

→ In-homogeneous wave with phase velocity → $\boxed{\omega/k_t = \frac{c}{n_1 \sin\theta}}$ and is greater than the velocity of light in a medium.

$\boxed{n_1 > n_2}$

Incident wave | **Reflected wave** | **Transmitted wave**



$\theta < \theta_c$: n_2 / n_1 (incidence medium) → Transmission (Refracted wave)

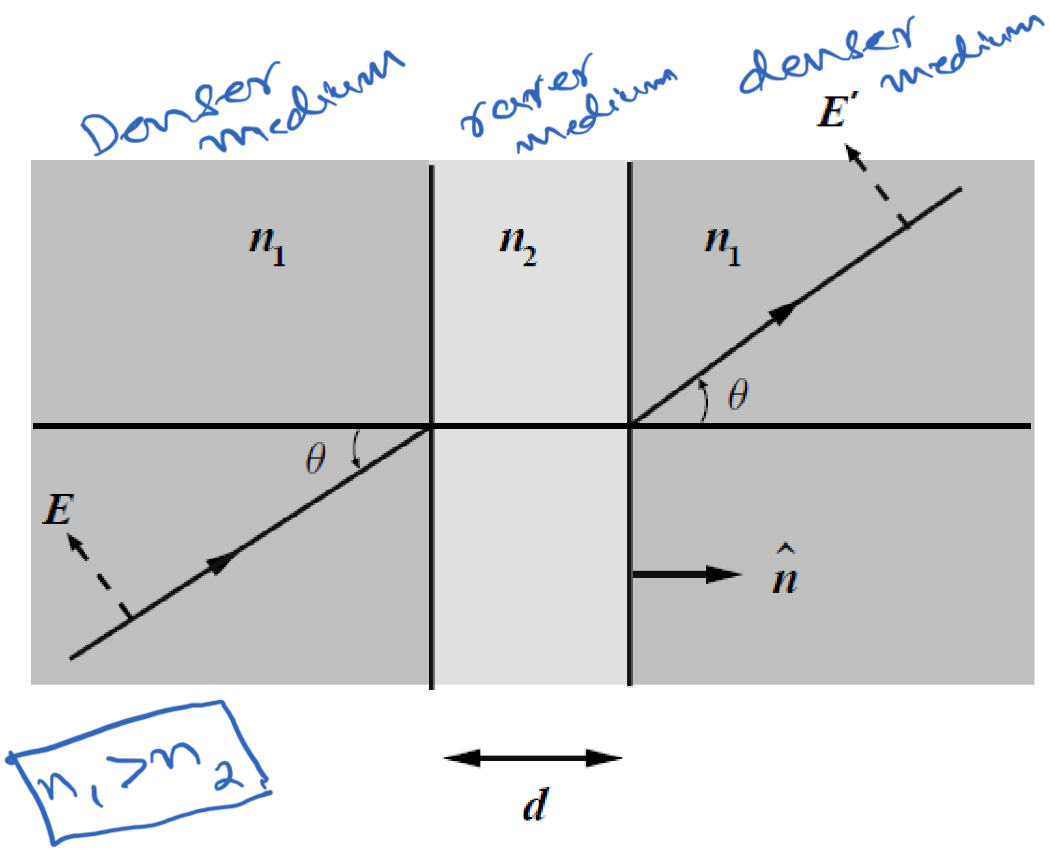
$$\mathbf{E}_{tr}(\mathbf{r}, t) = \mathrm{Re}\left(\mathbf{E}_{0,tr} e^{-\kappa_y y} e^{i(k_x x - \omega t)}\right)$$

$\theta = \theta_c$: n_2 / n_1 → Evanescent wave

$\theta > \theta_c$: n_2 / n_1 → TIR with Evanescent wave

Source: https://physics.stackexchange.com/

FRUSTRATED TOTAL INTERNAL REFLECTIONS



$\boxed{n_1 > n_2}$

Geometry to frustrate total internal reflection $n_2 < n_1$.

depth of penetration
$\delta << \lambda_2$
(fraction of wavelength)

→ In the TIR condition ($\theta > \theta_c$) the evanescent wave that propagates along the interface with amplitude decay into the second medium (n_2) will not have the energy flowing into the second medium

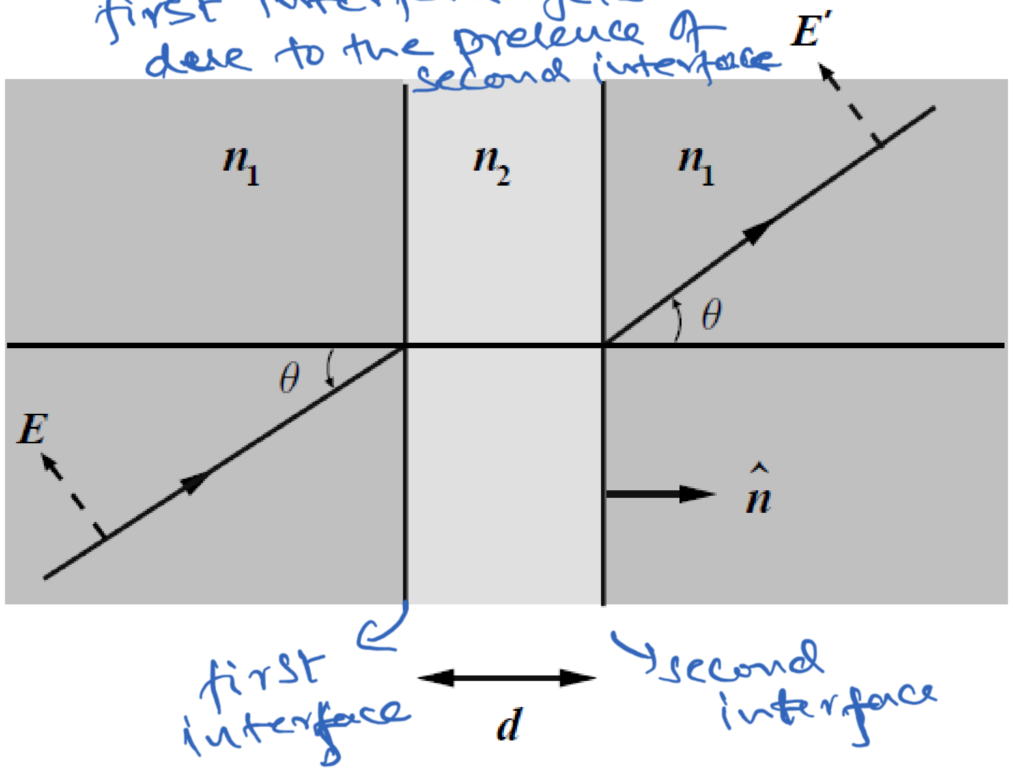
→ To investigate this we will check what is happening in the neighbourhood of the interface.

$d \to$ thickness of slab (n_2) $\gtrsim \delta$

Case I :- If 'd' is made sufficiently large ($d >> \delta$), then the transmitted wave after travelling a short distance (δ) in second medium (n_2) will reflect and re-enters the first medium but with shifted position.
→ TIR

FRUSTRATED TOTAL INTERNAL REFLECTIONS

→ Boundary conditions of first interface gets modified due to the presence of second interface



Geometry to frustrate total internal reflection $n_2 < n_1$.

$$t^2 + r^2 = 1$$

$t \rightarrow$ transmission coefficient

$r \rightarrow$ reflection coefficient,

$\phi_0 \rightarrow$ phase change after reflection

→ It is possible to frustrate this TIR (reduced TIR) by reducing the thickness (d) of the sandwiched medium (n_2)

→ For a light (plane polarized) of amplitude E entering the first interface (b/w n_1 & n_2), the amplitude of the wave entering the second interace (n_2 & n_1) is

$$E' = t\,t'\,e^{-d/\delta}\,E$$

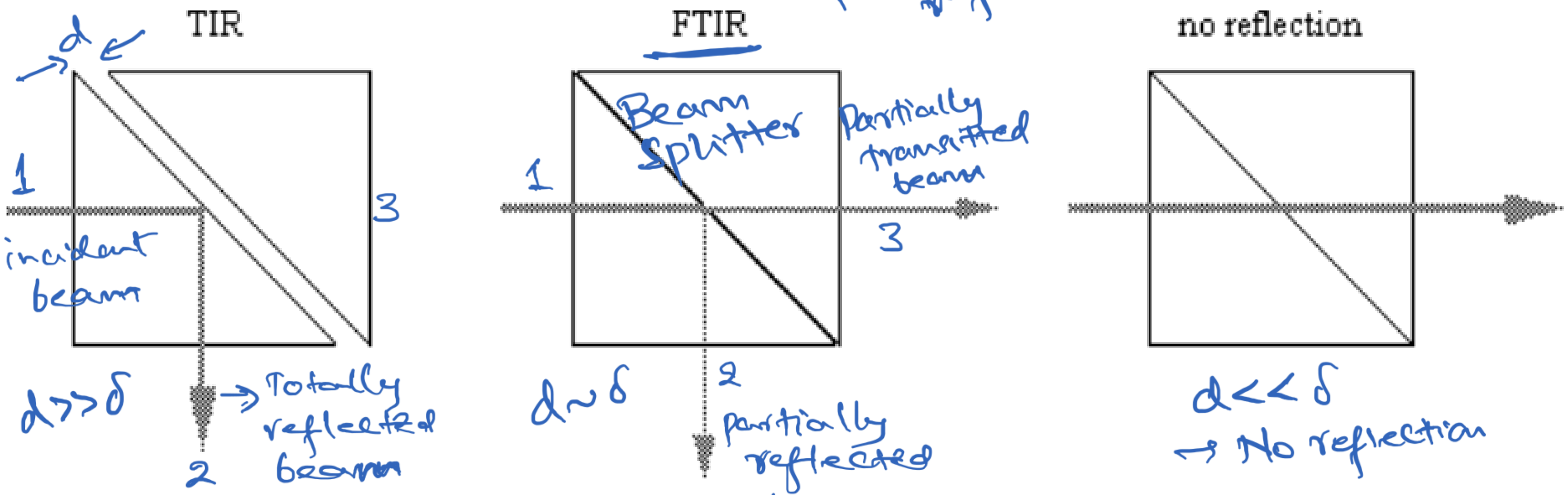
t & t' are the transmission coefficients at first and second interfaces, respectively

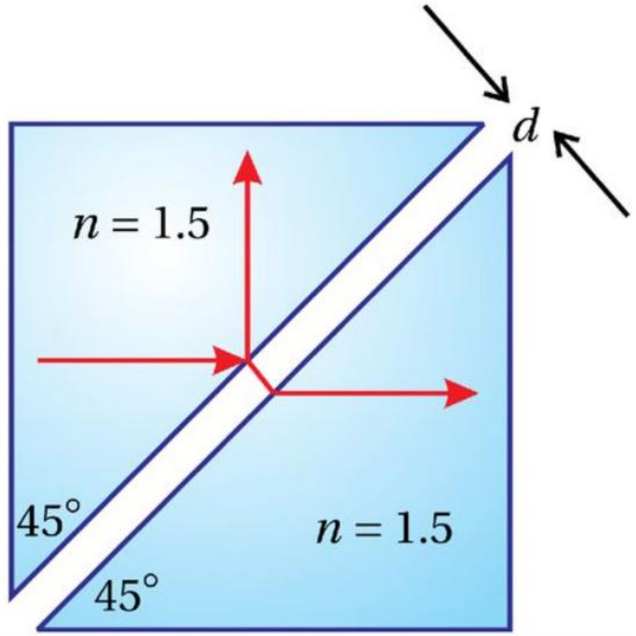
$$E' = (1 - r^2)\,e^{-d/\delta}\,E$$

$$\boxed{E' = \left(1 - e^{-i4\phi_0}\right)e^{-d/\delta}\,E}$$

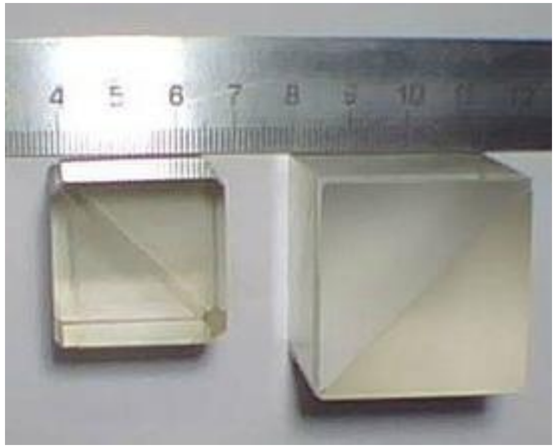
BEAM SPLITTER

→ Frustrated Total internal reflection



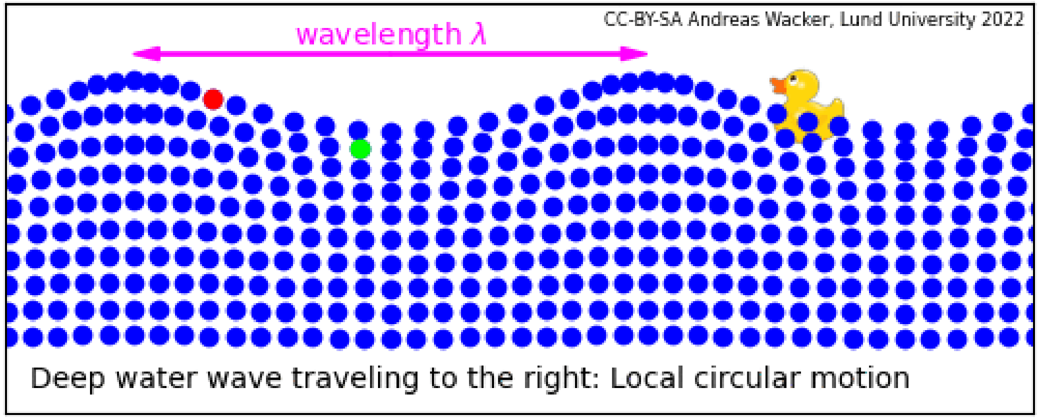


$$T_s^{tot} = \frac{n_2 \cos\theta_2}{n_0 \cos\theta_0} \left| t_s^{tot} \right|^2$$



The thickness of the resin layer is adjusted such that (for a certain wavelength) half of the light incident through one "port" (i.e., face of the cube) is reflected and the other half is transmitted due to FTIR (frustrated total internal reflection)

WAVE PROPAGATION AND WAVE EQUATIONS

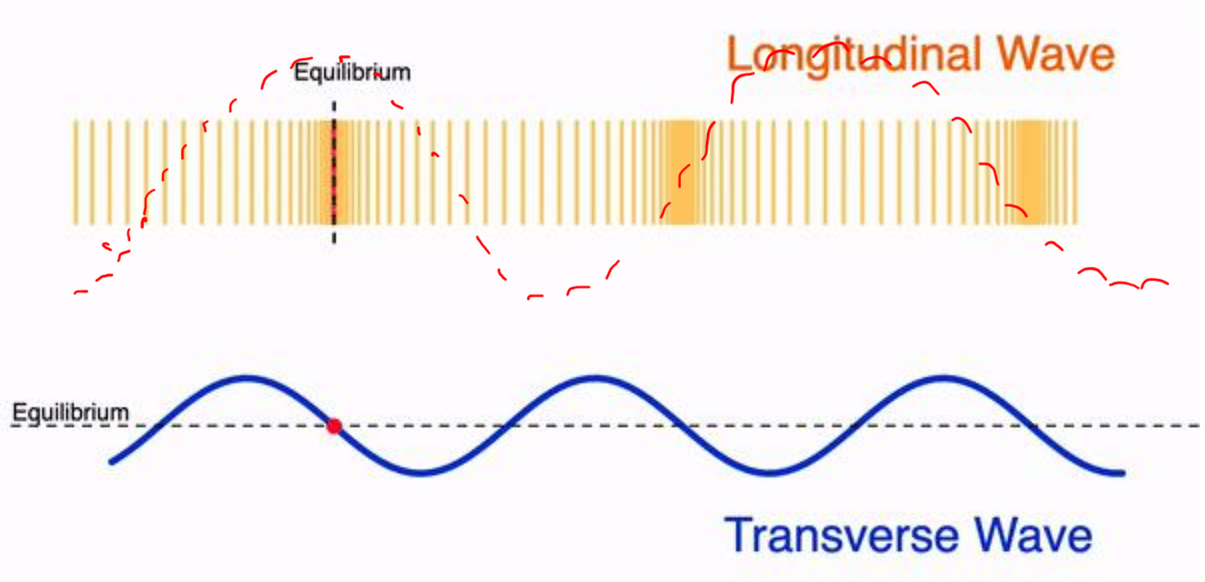


General form of wave equations

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$$

$$y(x,t) = a\cos(kx \pm \omega t + \varphi)$$

$$y(x,t) = a\, e^{i(kx - \omega t + \varphi)}$$

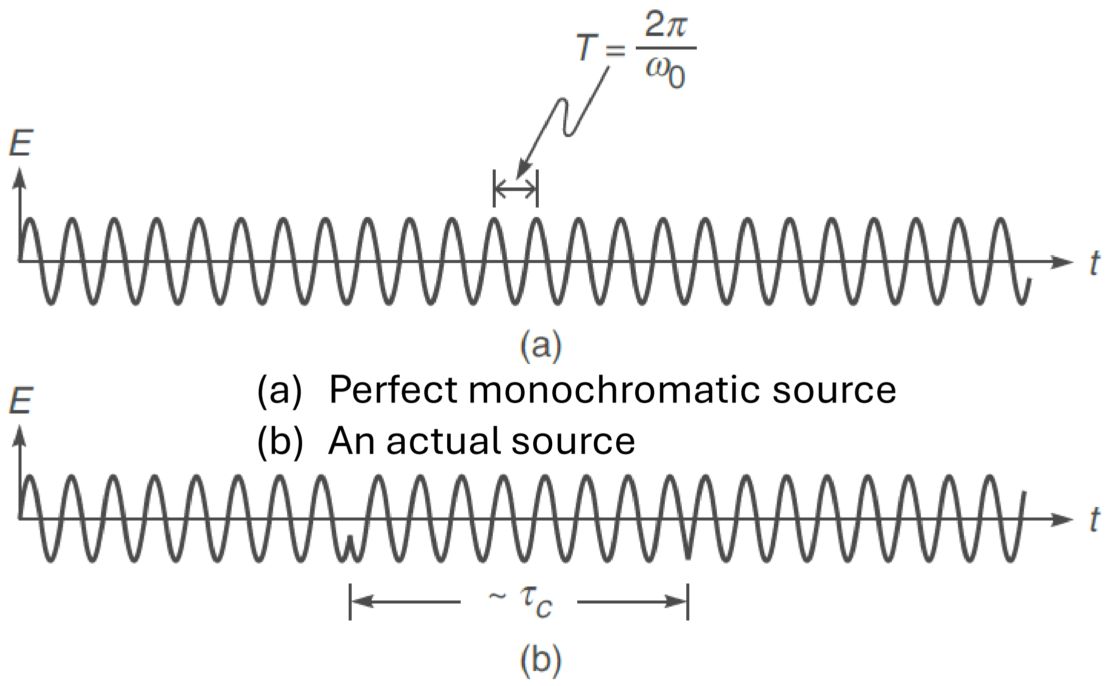


→ Displacement of waves is parallel to the propagation direction of wave

→ Displacement of waves is perpendicular to the propagation direction of wave

COHERENCE OF WAVES

→ Expresses the potential of two waves to interfere.



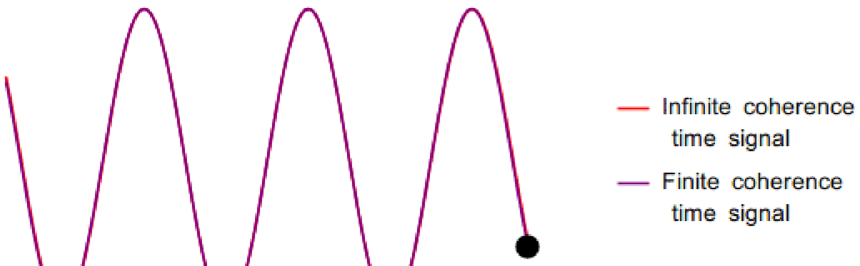
(a) Perfect monochromatic source
(b) An actual source

$$y(x,t) = a\cos(kx \pm \omega t + \varphi)$$

→ monochromatic wave with ~~the~~ temporal period $T = \frac{2\pi}{\omega_0}$

→ In practical situations we don't get perfect monochromatic behaviour in waves.

→ The electric field remains sinusoidal for some time window given by the coherence time (τ_c)



Time/Temporal coherence

→ within τ_c, the Electric field oscillations have definite phase relationship

→ The wave is said to be coherent for time τ_c

$\Delta t > \tau_c$ → incoherent (no phase relationship)

The corresponding length of the coherent wave train is given by $L_c = c\,\tau_c$ → <u>Coherence length</u>

→ The finiteness in coherence is due to imperfectness in source like collisions, finite transition line widths, temperature, ...

→ Coherence times for lasers are of the order of 'ns' to 'ms' and the corresponding coherence length ~ few cms to kms

→ ~~The~~, practically, light sources are partially coherent with finite τ_c & L_c

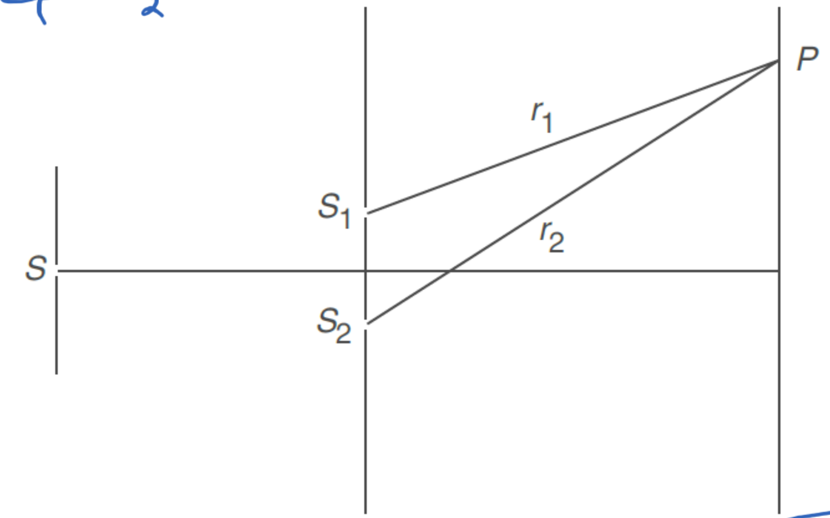
→ The amount of coherence can be measured or quantified by the interference visibility, and degree of interference.

QUANTIFICATION OF COHERENCE

Young's double slit example → Coherence time & length

$I = I_1 + I_2 + 2\sqrt{I_1 I_2}\cos(\epsilon_1 - \epsilon_2)$

$\epsilon_1 - \epsilon_2 = 2n\pi$ → constructive



$\epsilon_1 - \epsilon_2 = (2n+1)\pi$

destructive interference

S_1 and S_2 are sources (secondary wavelets)

r_1 and r_2 → distances of propagation of waves S_1P & S_2P respectively

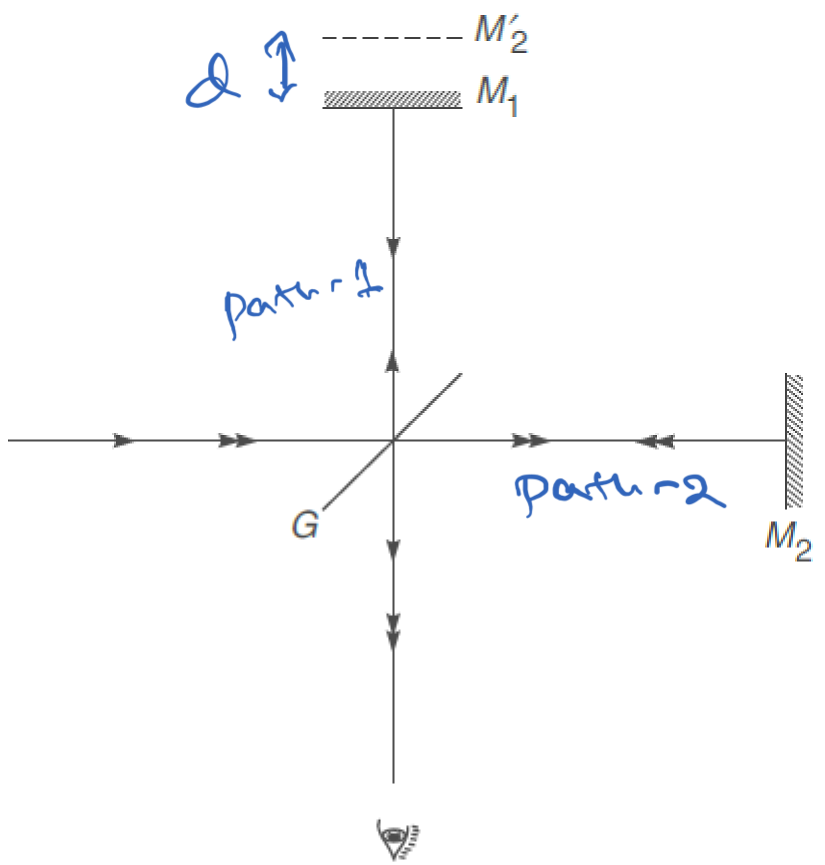
The interference pattern at P at time 't' is due to the superposition of wave S_1P and S_2P at time $t - r_1/c$ and $t - r_2/c$ respectively,

If $\frac{r_1 - r_2}{c} \ll \tau_c$ → the waves S_1P & S_2P will have definite phase relationship → Results in interference

$(r_1 - r_2)$ → path difference

If $\frac{r_1 - r_2}{c} \gg \tau_c$ → No fixed phase and hence No interference

QUANTIFICATION OF COHERENCE



Michelson's interferometer

the difference in the path length in path 1 & path-2 is '2d'

Corresponding time difference b/n the paths is '$2d/c$'

The light is said to be coherent if $2d/c < \tau_c$ ⇒ observe the interference pattern

if $2d/c \gg \tau_c$ ⇒ 'no' phase correlations and hence no interference pattern (reduced interference contrast)

THE LINEWIDTH → Temporal coherence

→ Using Michelson's interferometer one can determine both spatial coherence (L_c) and temporal coherence (τ_c)

→ Condition for interference is $n\lambda = 2d$ (→ order for fringes)
for two nearby wavelengths λ_1 & λ_2 & $n=1$
the destructive interference will occur when

$$\frac{2d}{\lambda_2} - \frac{2d}{\lambda_1} = \frac{1}{2}$$

$$\Rightarrow \quad 2d = \frac{\lambda_1 \lambda_2}{2(\lambda_1 - \lambda_2)} \sim \frac{\lambda^2}{2(\lambda_1 - \lambda_2)}$$

$$2d \sim \lambda^2 / \Delta\lambda$$

→ this is the minimum path length for which the complete destructive interference occurs

Therefore for $2d \geq \lambda^2/\Delta\lambda$ → the contrast of the interference fringes will be extremely poor.

The spectral width for observing the interference would be $\Delta\lambda \sim \lambda^2/2d = \lambda^2/L_c$

$$\boxed{\Delta\lambda = \frac{\lambda^2}{c\tau_c}}$$

L_c → Coherence length

τ_c → Coherence time

→ We can determine both the temporal and spatial coherence using the Michelson interferometer.

→ For monochromatic wave ($\Delta\lambda \approx 0$), the spatial coherence length $L_c \sim$ infinite.

For example:-

Helium - Neon (He-Ne) laser $\lambda \simeq 632\,nm$.
& $\Delta\lambda \sim 0.001\,nm$

$$\Delta\lambda = \frac{\lambda^2}{L_c}$$

has coherence length $L_c \sim$ 100's of meters or kms

where as, the LEDs, characterized by the $\Delta\lambda \simeq 50\,nm$, have very short coherence lengths, $L_c \sim \mu m - mm$ range

THE SPATIAL COHERENCE

→ It depends on the physical size of the light field

→ A point light source, the has the physical dimension less than the mean wavelength of the emitted light, possesses a high degree of spatial coherence, irrespective of it emission bandwidth.

→ The extended sources do not exhibit good spatial coherence, as they radiate independently and therefore are mutually incoherent,

→ spatial coherence light determines how far the two points on a plane wave are correlated in phase.

→ spatial and temporal coherence can be determined by the interferometric visibility of the fringes (ex: Young's double slit or Michelson stellar interferometer (spatial) and Michelson interferometer (Temporal))

Interferometric visibility (V) of the fringes

$$V = \frac{I_{max} - I_{min}}{I_{max} + I_{min}}$$

degree of complex coherence

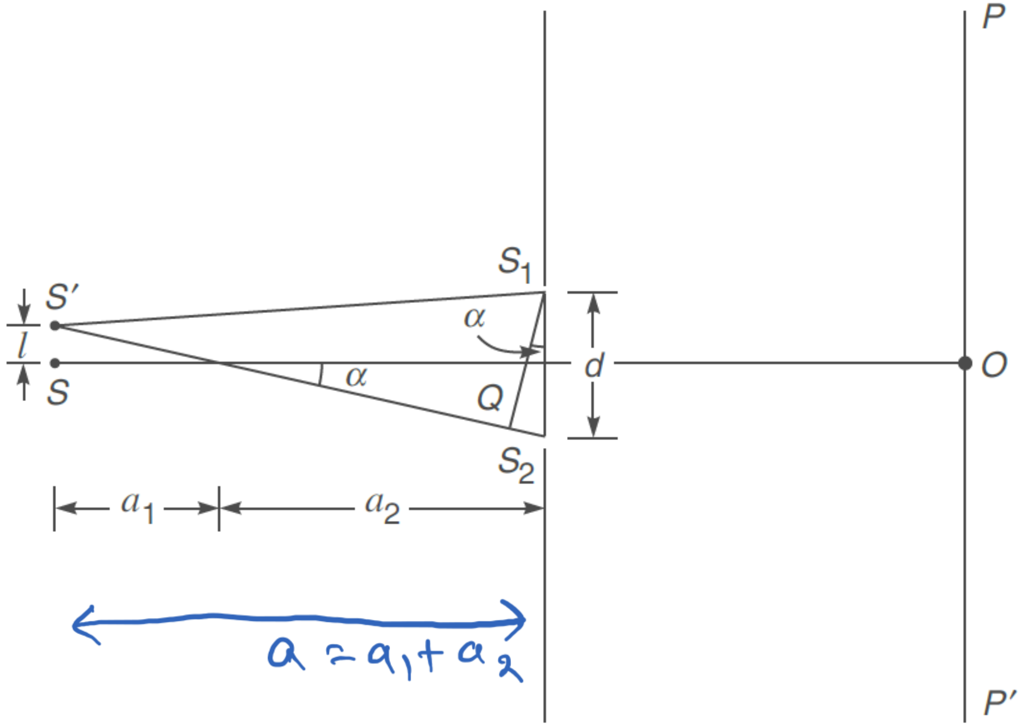
where, $I_{max} = I_1 + I_2 + 2\sqrt{I_1 I_2}\, \gamma_{12}$

$I_{min} = I_1 + I_2 - 2\sqrt{I_1 I_2}\, \gamma_{12}$

THE SPATIAL COHERENCE

→ Associated to the finite dimension of the source.

Can be quantified using Young's interference expt.



Sources 'S' and 'S'' donot possess definite phase relationship → The interference formed at 'O' would be superposition of both paths due to S and S'.

The interference due to S' will be shifted by an amount

$$S'S_2 - S'S_1 = \lambda/2$$

→ results in minima @ point 'O'

S → monochromatic point source

$SS_1 = SS_2$ ⇒ maxima at point 'O' on PP' screen.

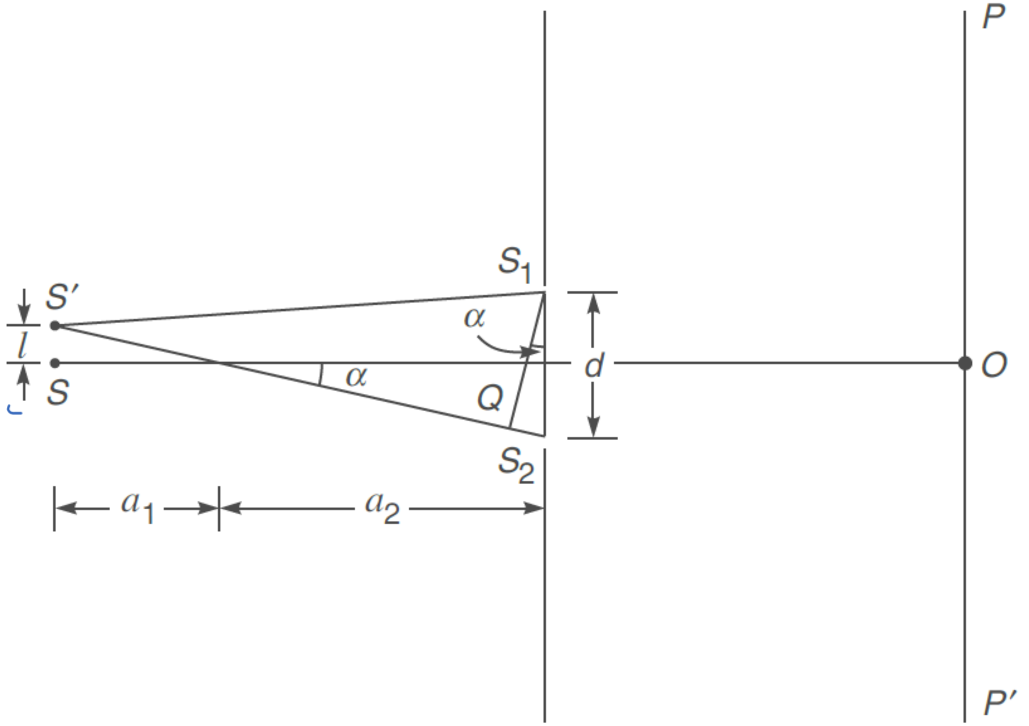
S' → monochromatic point source at a distance 'l' from S.

For the interference to disappear the condition is

$$\lambda/2 = S'S_2 - S'S_1 = \frac{ld}{a}$$

or $l = \dfrac{\lambda a}{2d}$ → extention length of source

THE SPATIAL COHERENCE



$l \rightarrow$ source extension length

$d \rightarrow$ slit seperation

$a \rightarrow$ source - slit distance

For an extended incoherent source, the interference fringes of good contrast will be observed only, if the source dimension

$$l \ll \frac{\lambda a}{d}$$

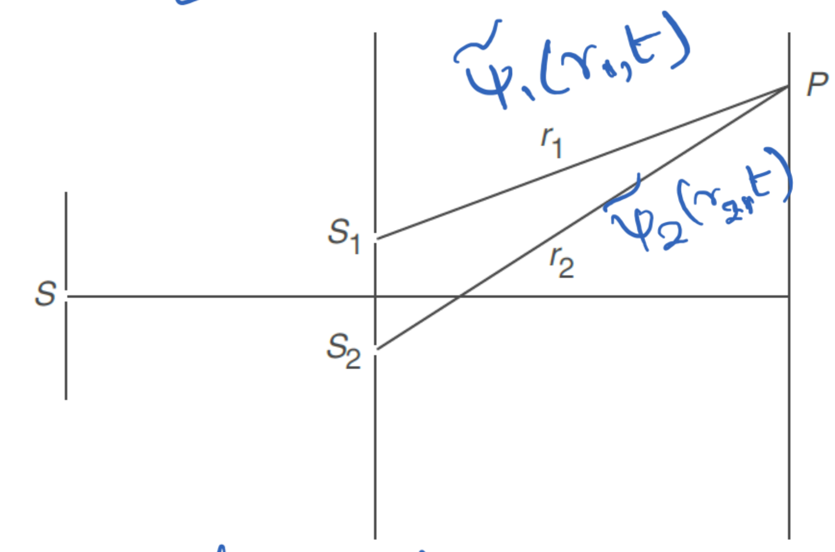
If $l = \frac{\lambda a}{d}$ $\rightarrow$ the point on source at a distance $\frac{\lambda a}{2d}$ produces fringes which are shifted by half the fringe width therefore no-interference will be observed.

$$\boxed{l_w = \frac{\lambda a}{d}} \rightarrow \text{Lateral coherence width}$$

This gives the distance over which the beam may be assumed to be spatially coherent.

Complex degree of coherence and Fringe Visibility

Young's double slit expt



$S_1P = \left(t - \frac{r_1}{c}\right)$

$S_2P = \left(t - \frac{r_2}{c}\right)$

$\tilde{\psi}_1$ & $\tilde{\psi}_2$ are complex wave functions

The resultant amplitude (displacement) @ point 'P' is given by

$$\tilde{\psi} = \tilde{\psi}_1(P,t) + \tilde{\psi}_2(P,t)$$

Now the intensity at point 'P' is

$$|\psi|^2 = \psi_1^*\psi_1 + \psi_2^*\psi_2 + \psi_1^*\psi_2 + \psi_2^*\psi_1$$

$$= |\psi_1|^2 + |\psi_2|^2 + 2\,\mathrm{Re}(\psi_1^*\psi_2)$$

Intensity $I = \langle|\psi(P,t)|\rangle^2 \rightarrow$ square of time avg. of amplitudes

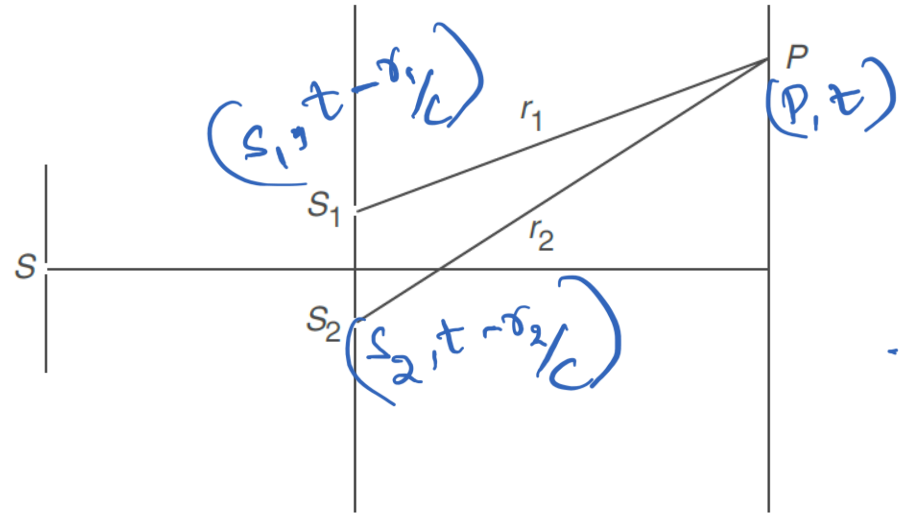
$$\therefore \boxed{I = I_1 + I_2 + 2\sqrt{I_1 I_2}\,\mathrm{Re}\,\gamma_{12}}$$

where

$$\boxed{\gamma_{12} = \frac{\langle \psi_1^*(P,t)\,\psi_2(P,t)\rangle}{\left[\langle|\psi_1(P,t)|^2\rangle\langle|\psi_2(P,t)|^2\rangle\right]^{1/2}}}$$

Complex degree of coherence.

Complex degree of coherence and Fringe Visibility



The $\tilde{\Psi}_1(P,t) \rightarrow$ wave from S_1 at $(t - r_1/c)$

$\therefore \tilde{\Psi}_1(P,t) \propto \tilde{\Psi}(S_1, t - r_1/c)$

$|||^{ly}$ $\tilde{\Psi}_2(P,t) \propto \tilde{\Psi}(S_2, t - r_2/c)$

$\therefore$ The complex degree of coherence

$$\gamma_{12} = \frac{\langle \psi^*(S_1, t - r_1/c)\, \psi(S_2, t - r_2/c) \rangle}{\sqrt{\left[\langle |\psi(S_1, t - r_1/c)|^2 \rangle \langle |\psi(S_2, t - r_2/c)|^2 \rangle\right]}}$$

$\langle \ \rangle \rightarrow$ time averaging

time delay $\rightarrow \tau = \dfrac{r_2 - r_1}{c}$ $\rightarrow$ Then, $\gamma_{12} = \dfrac{\langle \psi^*(S_1, t+\tau)\, \psi(S_2, t) \rangle}{\sqrt{\left[\langle |\psi^*(S_1, t+\tau)|^2 \rangle \langle |\psi(S_2, t)|^2 \rangle\right]}}$

$S, (t + \tau = 0) \rightarrow$ spatial coherence

$S = 0, (t + \tau) \rightarrow$ Temporal coherence

Complex degree of coherence and Fringe Visibility

→ If S_1 and S_2 are of negligible spatial dimensions, and S_1 and S_2 are equidistant from S, then

$$\psi(S_1, t) = \psi(S_2, t) = \psi(t)$$ ⇒ Temporal coherence

$$\therefore \quad \boxed{\gamma_{12}(\tau) = \frac{\langle \psi^*(t+\tau)\,\psi(t)\rangle}{\langle |\psi(t)|^2\rangle}}$$

If $\psi(t) = A(t)\, e^{-i(\omega t + \phi(t))}$

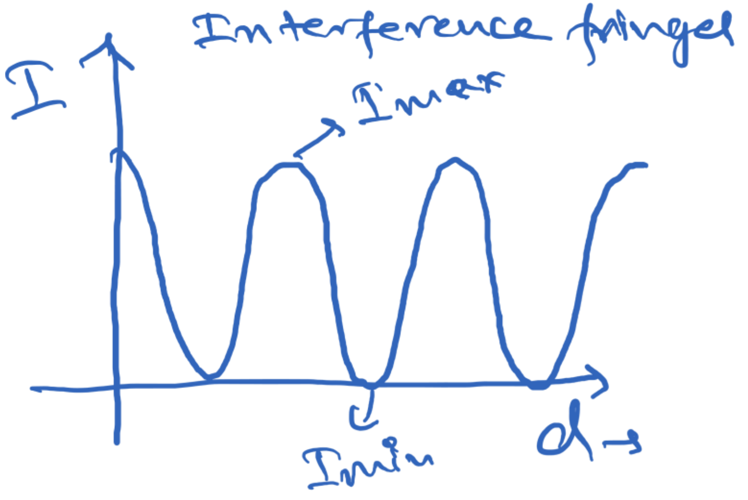
Then for a perfectly monochromatic wave, $A(t)$ & $\phi(t)$ are constant,

$$\Rightarrow \quad \gamma_{12}(\tau) = \frac{A^2 e^{i\omega\tau}}{A^2} = e^{i\omega\tau}$$ → purely monochromatic

In general $\boxed{\gamma_{12}(\tau) = |\gamma_{12}|\, e^{i(\omega\tau + \beta)}}$

$|\gamma_{12}|$ and β are amplitude and phase, respectively.

Complex degree of coherence and Fringe Visibility



$|\gamma_{12}| \rightarrow$ magnitude of degree of coherence

Fringe visibility (V)

$$V = \frac{I_{max} - I_{min}}{I_{max} + I_{min}}$$

where $\quad I = I_1 + I_2 + 2\sqrt{I_1 I_2}\,\gamma_{12}(\tau)$

$$= I_1 + I_2 + 2\sqrt{I_1 I_2}\,|\gamma_{12}|\cos(\omega\tau + \beta)$$

$$I_{max} = I_1 + I_2 + 2\sqrt{I_1 I_2}\,|\gamma_{12}|$$

$$I_{min} = I_1 + I_2 - 2\sqrt{I_1 I_2}\,|\gamma_{12}|$$

Then

$$\boxed{V = \frac{I_{max} - I_{min}}{I_{max} + I_{min}} = \frac{2\sqrt{I_1 I_2}}{I_1 + I_2}\,|\gamma_{12}|}$$

For perfectly monochromatic wave $I_1 = I_2 \Rightarrow V = |\gamma_{12}|$

and $|\gamma_{12}| = 1 \Rightarrow$ purely coherent wave (light)

Complex degree of coherence and Fringe Visibility

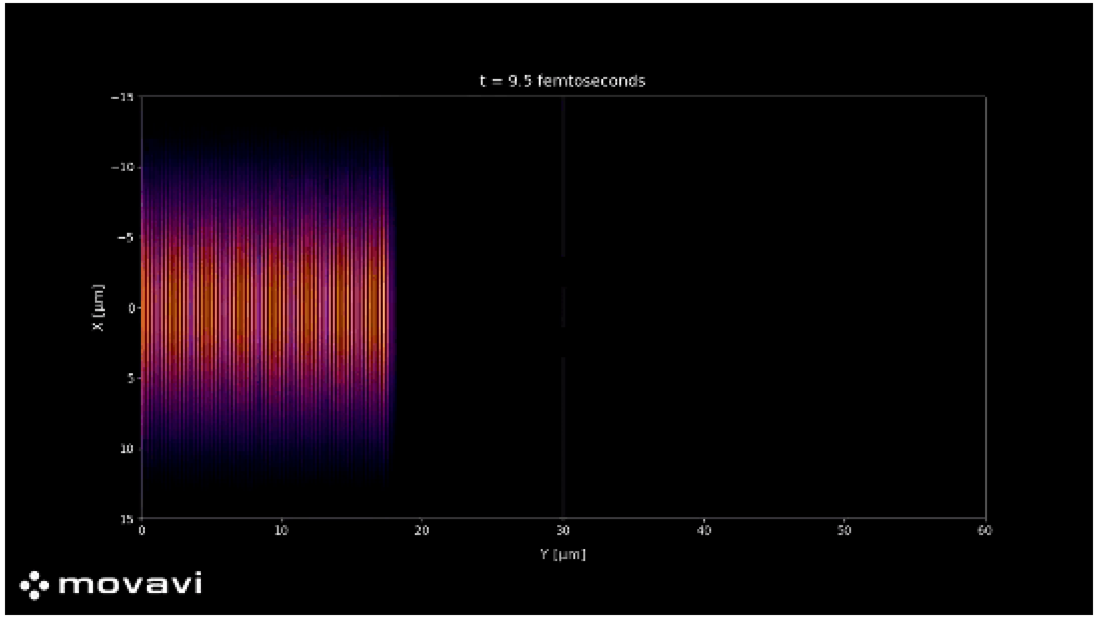
The visibility (V) (also called contrast) of the fringes is a direct measure of $|\gamma_{12}|$ (degree of coherence)

For quantifying the degree of temporal coherence of the beam, $\tau \ll \tau_c$, the $|\gamma_{12}|$ is very close to '1', and the contrast of the fringes will be very good. $|\gamma_{12}| \approx 1$

if $\tau \gg \tau_c$, then $|\gamma_{12}|$ will be close to '0', then the interference fringes contrast will be very poor. $|\gamma_{12}| \approx 0$

$\therefore$ $|\gamma_{12}|$ value decides/determines the visibility of the fringes and hence the spatial and Temporal coherence of the beam.

The Spatial Coherence



The light intensity of the interfered fringes are averaged over a few picosecond time

→ Purely coherent wave

Interferometric visibility of the fringes for a highly coherent wave

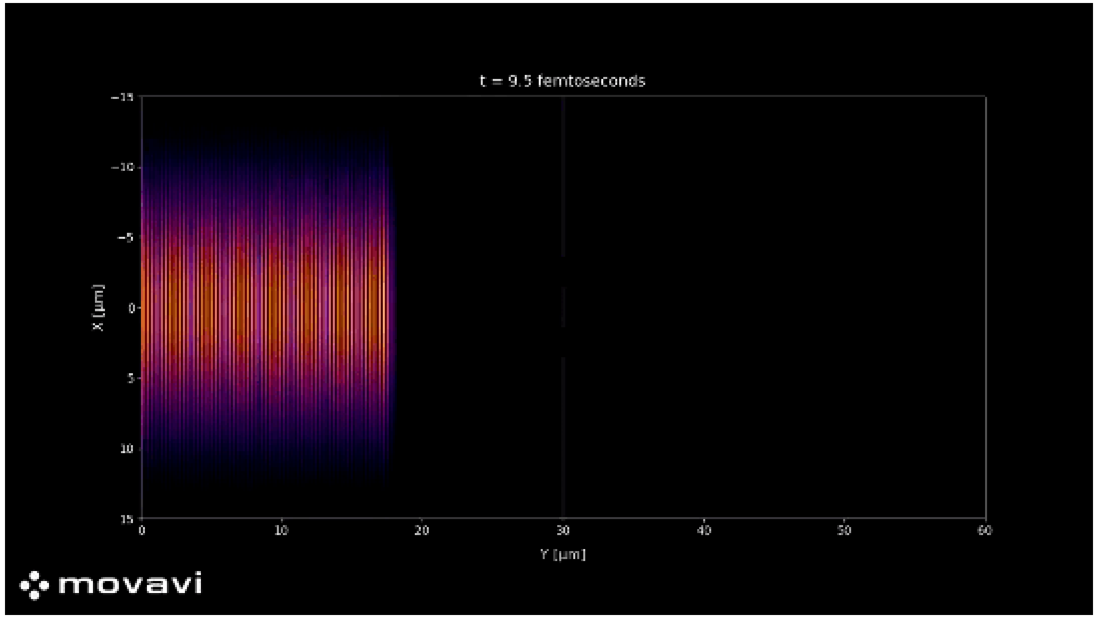
$$V = \frac{I_{max} - I_{min}}{I_{max} + I_{min}} \simeq 1$$

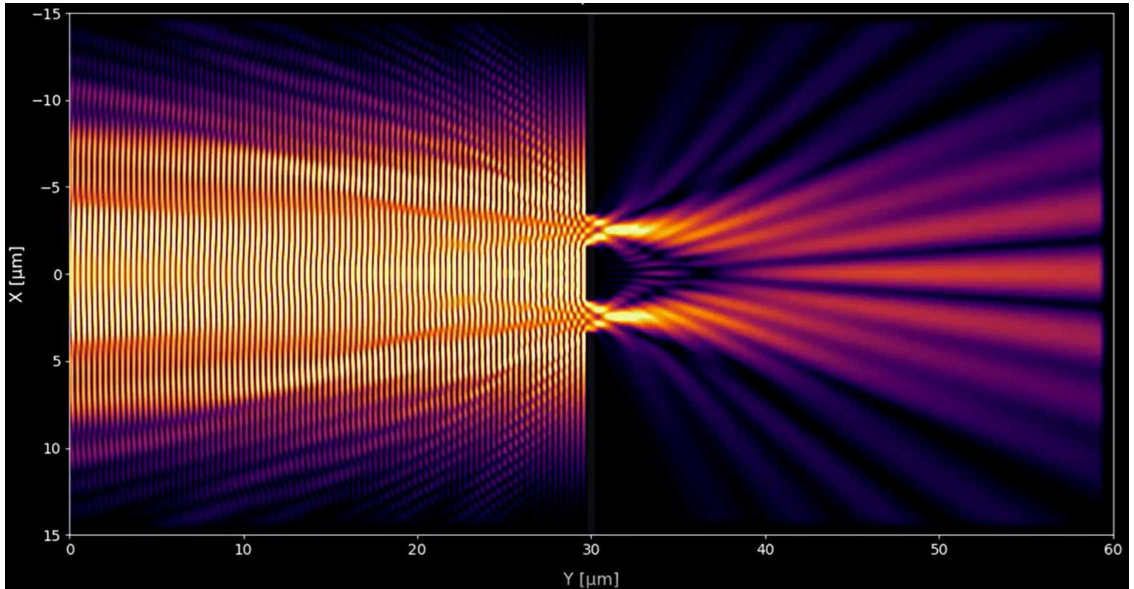
→ simulation showing the spatial interference and the visibility of the fringes in the young's double slit expt,

→ each wavefronts of the interfered light have definite (constant) phase, hence the spatial stability of the interfered fringes over the distance.

https://rafael-fuente.github.io/

The Spatial Coherence

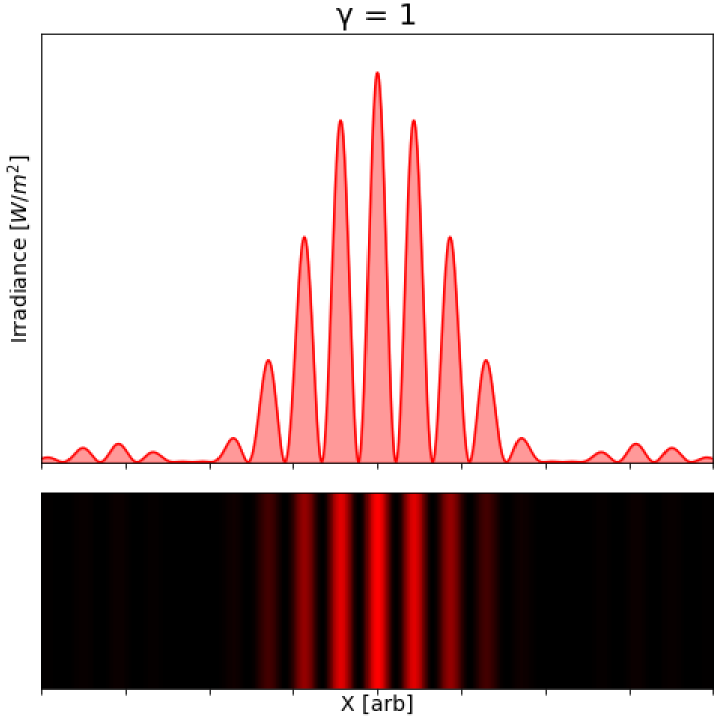




For a highly / purely coherent wave

$$V = \frac{I_{max} - I_{min}}{I_{max} + I_{min}} = 1$$

$\Rightarrow$ 100% visibility of the interfered fringes.



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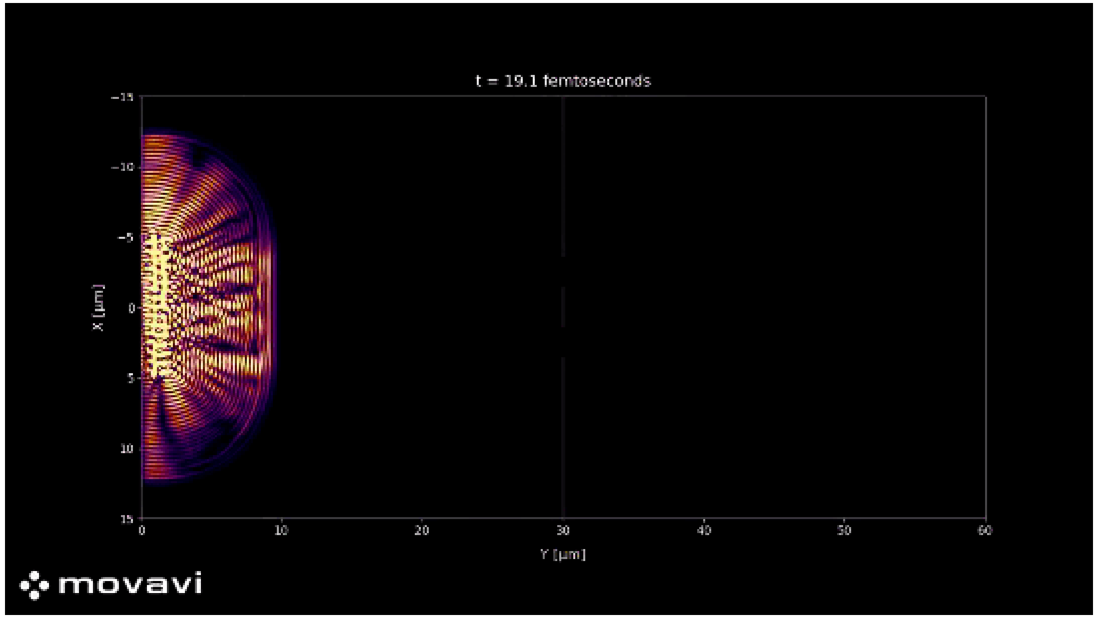
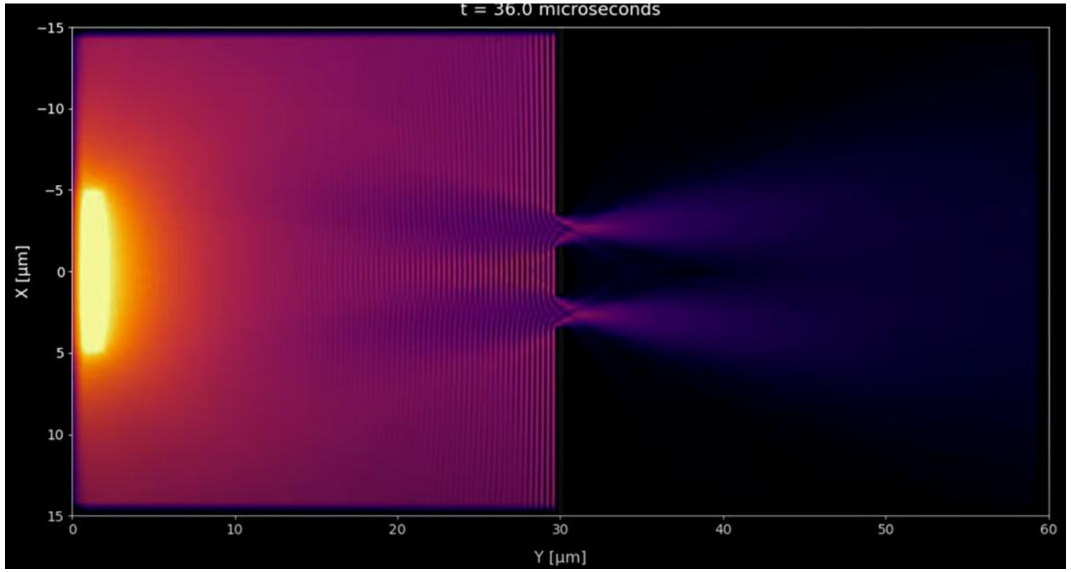
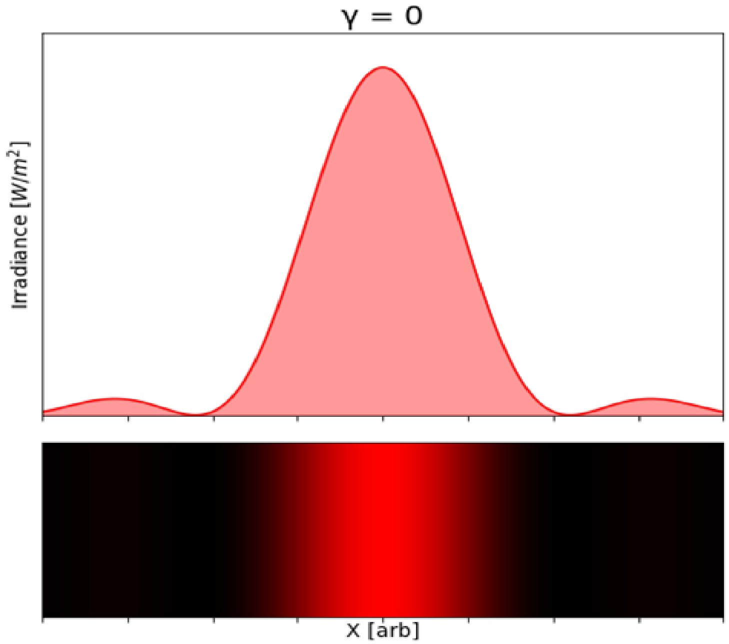
The Spatial Coherence





→ Incoherent light wave
→ Extended source
(No-phase correlations)
→ Fringe intensity averaged over milliseconds
→ Interferometric visibility of the fringes $\gamma = 0$
→ No interference!



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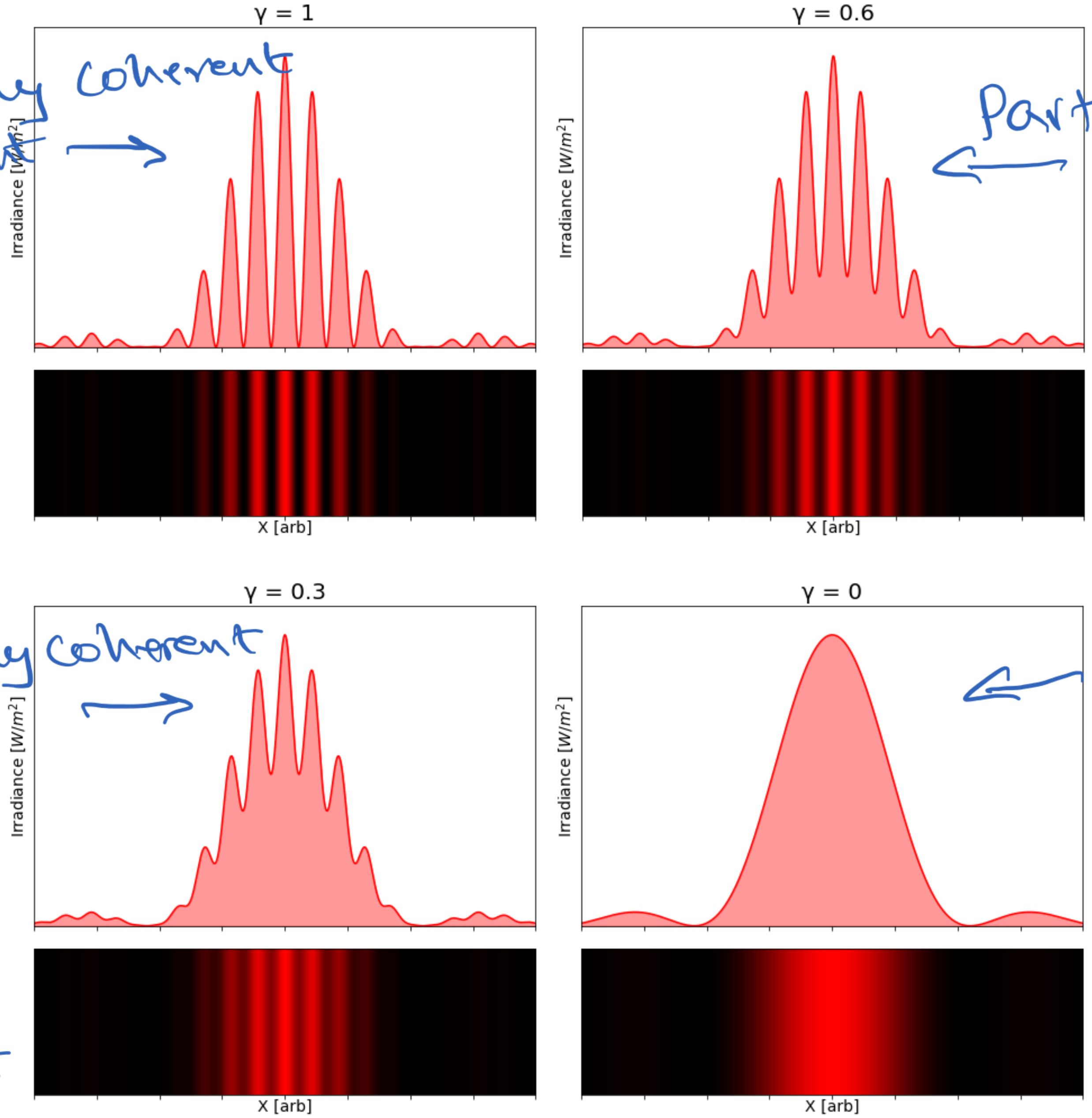
The Spatial Coherence

purely coherent light →

Partially coherent light

In practical cases, we always observe partially coherent light waves from the sources.



partially coherent light →

Incoherent light.

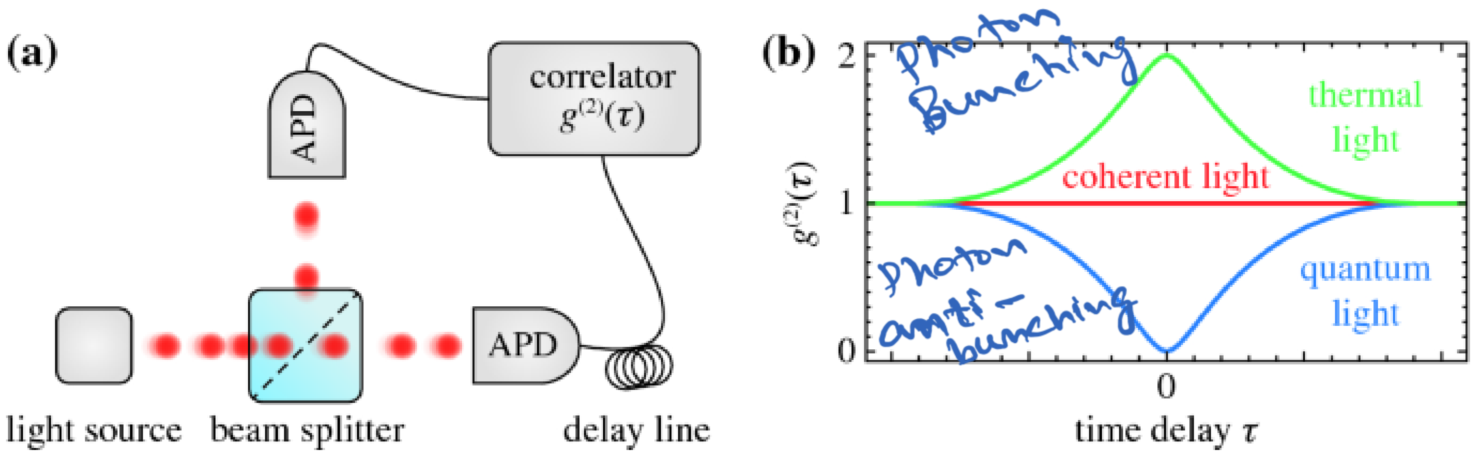
No-interference

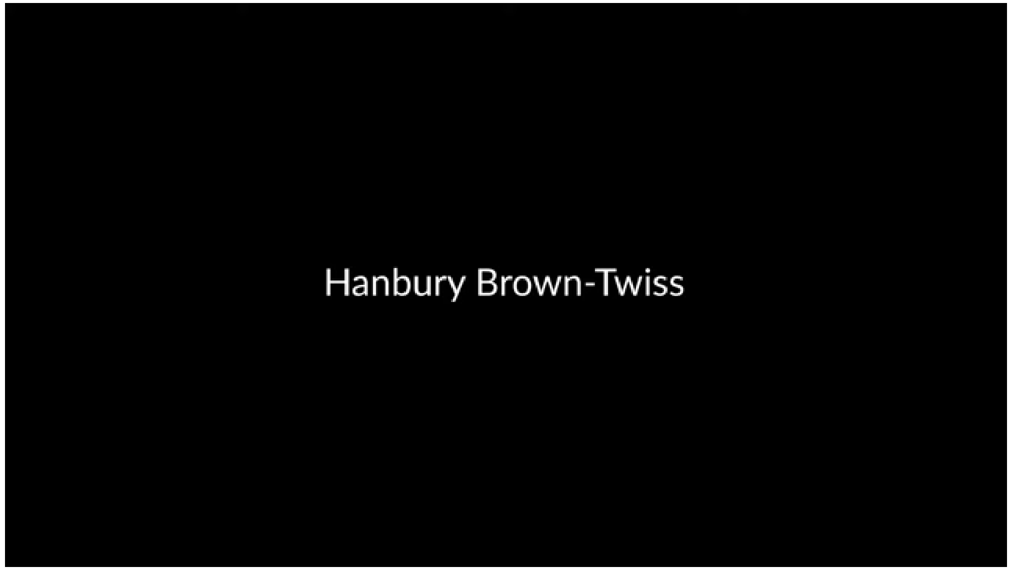
Fringe visibility
$$\boxed{0 \le \gamma \le 1}$$

if $0 < \gamma < 1$

→ partially coherent light.

The Temporal Coherence : Hanbury-Brown-Twiss Interferometer ($g^{(2)}$ correlations)





$$g^{(2)}(\tau) = \frac{\langle \psi^*(t+\tau)\,\psi(t)\rangle}{\langle |\psi(t)|^2 \rangle}$$

→ Second order correlation function [($g^{(2)}$) function]

$g^{(2)}(\tau) = 0 \Rightarrow$ No coherence (ideal single photons source)

$g^{(2)}(\tau) = 1 \Rightarrow$ coherent light (Laser) or fluorescent light

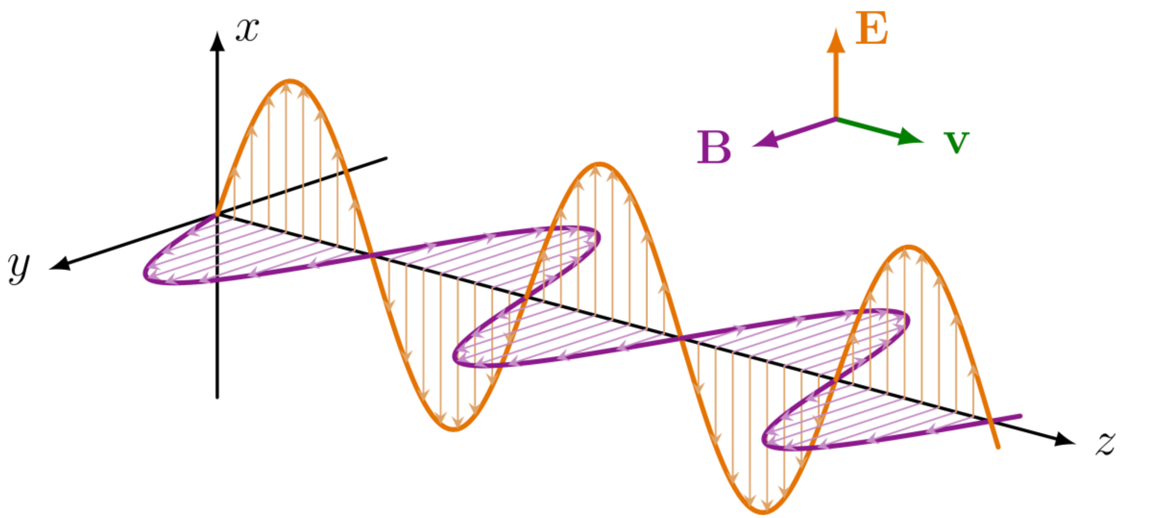
$0 \leq g^{(2)}(\tau) \leq 1$

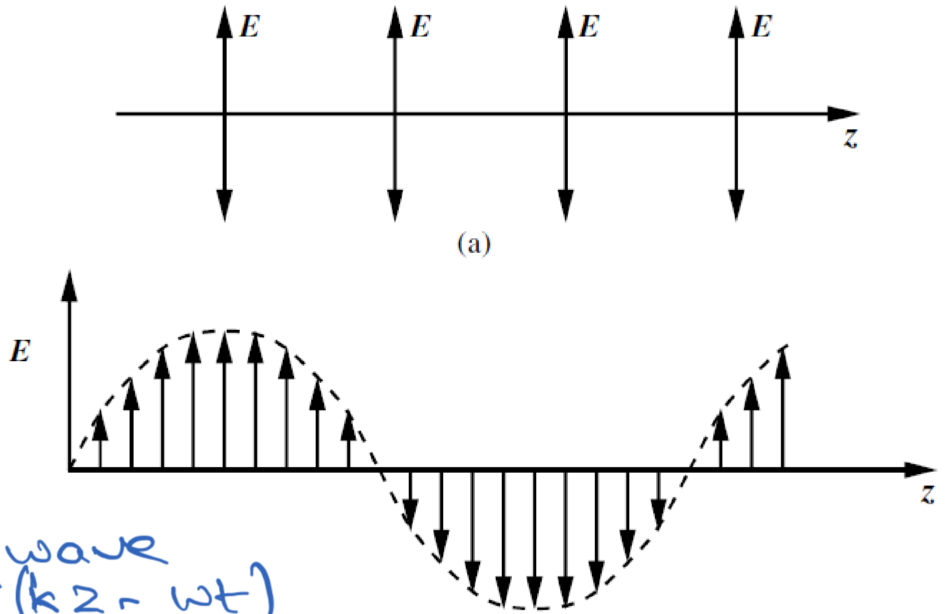
If $g^{(2)}(\tau) < 0.5$, it signifies the source is a single photon source

$g^{(2)} = 1 - \frac{1}{n}$; $n \rightarrow$ no. of Fock state

if $n = 1 \Rightarrow g^{(2)} = 0$
Pure single photon state

POLARIZATION OF LIGHT WAVES





A monochromatic plane wave

$$\vec{E}(\vec{r},t) = \vec{E_0}\, e^{i(kz - \omega t)}$$

$$\vec{B}(\vec{r},t) = \vec{B_0}\, e^{i(kz - \omega t)}$$

$z \rightarrow$ propagation direction of wave, in isotropic medium

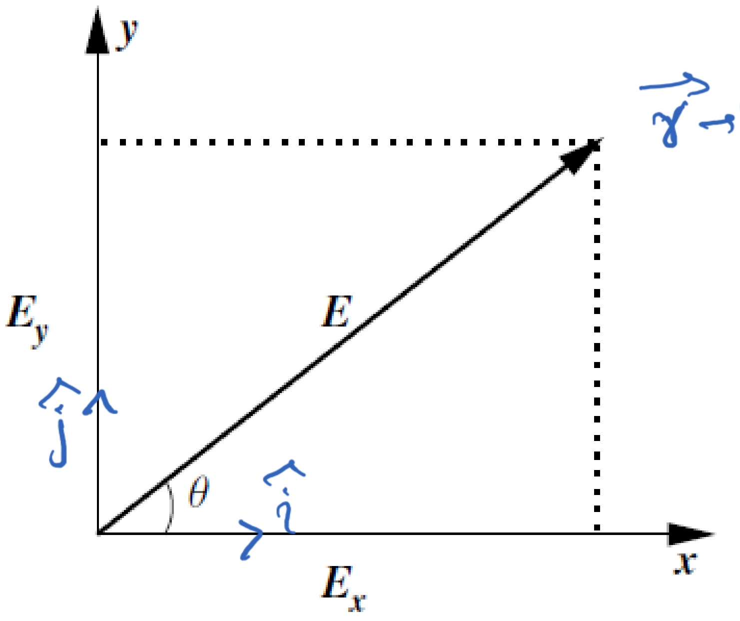
$\vec{E_0}$ & $\vec{B_0}$ are amplitude vectors ~~are~~ complex (in general)

$\rightarrow$ The direction of the electric field amplitude $\vec{E_0}$, is taken to represent (in general) the state of polarization of light $\rightarrow$ the transverse components of the electric field.

We can represent

$$\vec{E}(\vec{r},t) = \vec{E}(x,y,z,t) = E_x\hat{i} + E_y\hat{j}$$

$$= (E_{0x}\hat{i} + E_{0y}\hat{j})\, e^{i(kz-\omega t)}$$



$\vec{r} \to (x,y,z)$

→ Light is linearly polarized if the field components E_x & E_y oscillate in phase or out of phase by 180°

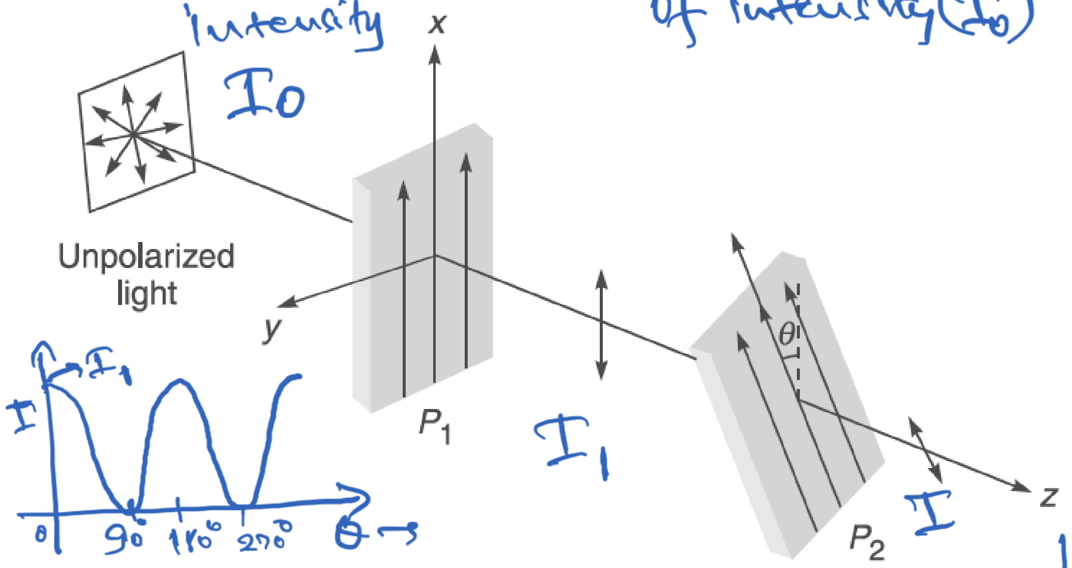
⇒ The field components E_x & E_y attain extremum values at the same time

→ Light is horizontally polarized (E_H) if the vertical component $E_y = 0$ and vice versa.

→ General state of linear polarization (E_θ) occures when E_x and E_y are both non zero

$\theta \to$ angle of polarization $\qquad \theta = \tan^{-1}\left(E_{0y}/E_{0x}\right)$

→ Let us consider an unpolarized light of intensity (I_0) passing through a polarizer (P_1) having the polarization (transmission) axis along x-axis.



→ The transmitted light after (P_1) will be linearly polarized along x-axis, with intensity of light (I_1).

→ Now place another polariser (P_2) (analyser) with the transmission axis rotated at an angle θ w.r.t. x-axis

$\theta = 0^\circ \Rightarrow \cos\theta = 1$ (max. intensity $I = I_1$)

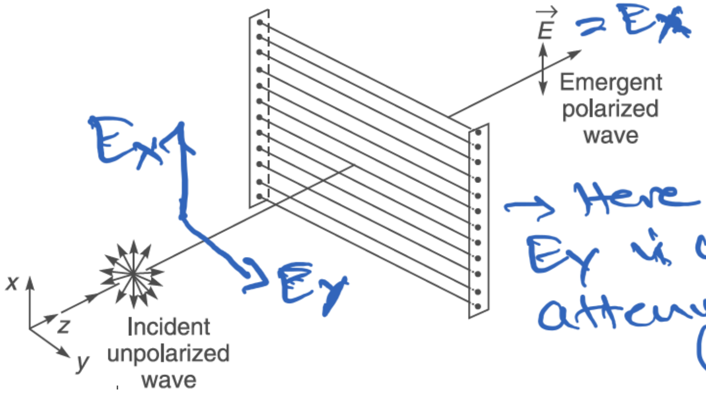
$\theta = 90^\circ \Rightarrow \cos\theta = 0$ (No transmission)

Then the intensity (I) after the analyser (P_2) is given by

$$\boxed{I = I_1 \cos^2\theta}$$ → Malus' law

→ Intensity of the transmitted linearly polarized light depends on the cosine of the angle b/w transmission axis and the direction of input polarization.

→ Types of polarizers (linear)

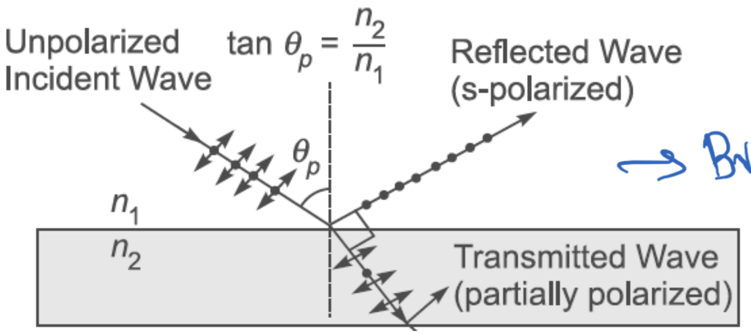


→ Here Ey is completely attenuated (absorbed)

Wire grid polarizers

→ made of metal wire gratings with spacing less than wave length of light (λ)

→ used mainly for Microwave / lower

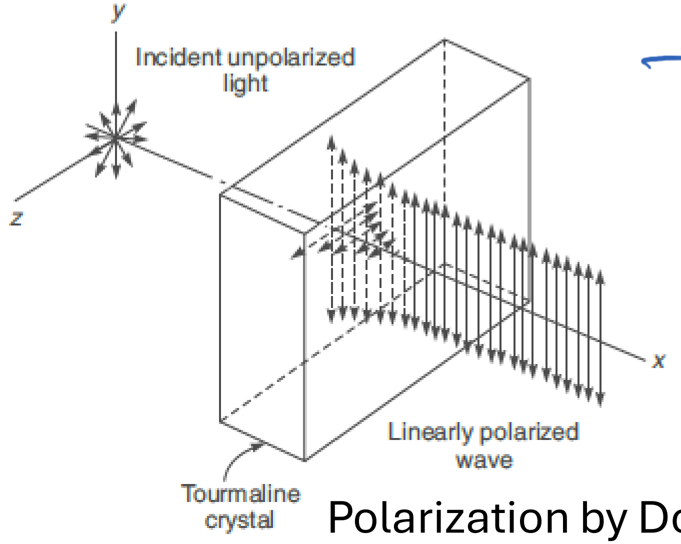


→ Brewster's angle

$\theta_r + \theta_t = 90^\circ$

Polarization by Reflection

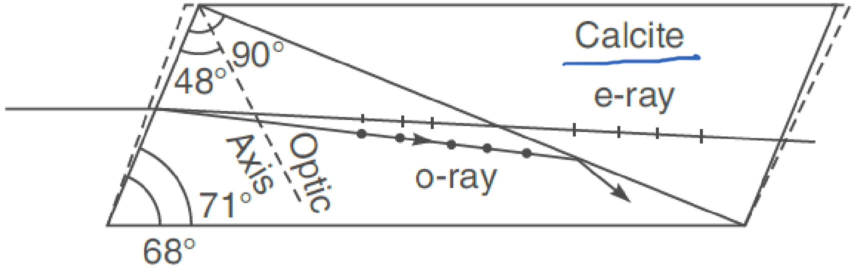
→ Sunglasses (polarized glasses)



→ Anisotropic crystal (birefringent) with different absorption coefficient of fast and slow axis

→ selective absorption → dichroism

Polarization by Double refraction



Polarization by TIR

→ Nicol prism

→ O-ray is total internal reflected by cutting the crystal in desired angles.

ELLIPTICAL AND CIRCULAR POLARIZATIONS

→ When light propagating in some materials (non-homogeneous, anisotropic), the oscillations of E_x and E_y components are generally not in phase.

Therefore, we can write $E_x = \mathcal{E}_x e^{i(kz - \omega t + \phi_x)}$

and $E_y = \mathcal{E}_y e^{i(kz - \omega t + \phi_y)}$

where $\mathcal{E}_x$ & $\mathcal{E}_y$ are amplitudes and are real.

For linearly polarized light $\phi_y - \phi_x = 0, \pm\pi, \ldots, \pm n\pi$

$n = 0, 1, 2, \ldots$

so that $$\frac{E_y}{E_x} = \frac{\mathcal{E}_y}{\mathcal{E}_x} \quad \text{for } \phi_y = \phi_x$$

and $$\frac{E_y}{E_x} = -\frac{\mathcal{E}_y}{\mathcal{E}_x} \quad \text{for } \phi_y = \phi_x \pm \pi$$

$$\Rightarrow \boxed{E_y = \pm \frac{\mathcal{E}_y}{\mathcal{E}_x} E_x}$$ → Requirement for linearly polarized light.

ELLIPTICAL AND CIRCULAR POLARIZATIONS

if $(\phi_y - \phi_x) = \phi_0$ → phase difference

Then we can write

$$E_x = \mathcal{E}_x \cos(kz - \omega t)$$
$$E_y = \mathcal{E}_y \cos(kz - \omega t + \phi_0)$$

$$\frac{E_x}{\mathcal{E}_x} = \cos(kz - \omega t) \quad \text{—(1)}$$

and

$$\frac{E_y}{\mathcal{E}_y} = \cos(kz - \omega t + \phi_0)$$
$$= \cos(kz - \omega t)\cos\phi_0 - \sin(kz - \omega t)\sin\phi_0 \quad \text{—(2)}$$

Multiplying eqn (1) by $\cos\phi_0$ and substracting it from eqn (2)

we get,

$$\frac{E_y}{\mathcal{E}_y} - \frac{E_x}{\mathcal{E}_x}\cos\phi_0 = -\sin(kz - \omega t)\sin\phi_0$$
$$= -\sin\phi_0\sqrt{1 - \cos^2(kz - \omega t)}$$

→ Squaring both sides, we get

ELLIPTICAL AND CIRCULAR POLARIZATIONS

$$\left(\frac{E_y}{\mathcal{E}_y}\right)^2 + \left(\frac{E_x}{\mathcal{E}_x}\right)^2 \cos^2\phi_0 - 2\left(\frac{E_y}{\mathcal{E}_y}\right)\left(\frac{E_x}{\mathcal{E}_x}\right)\cos\phi_0 = \left[1 - \cos^2(kz - \omega t)\right] \sin^2\phi_0$$

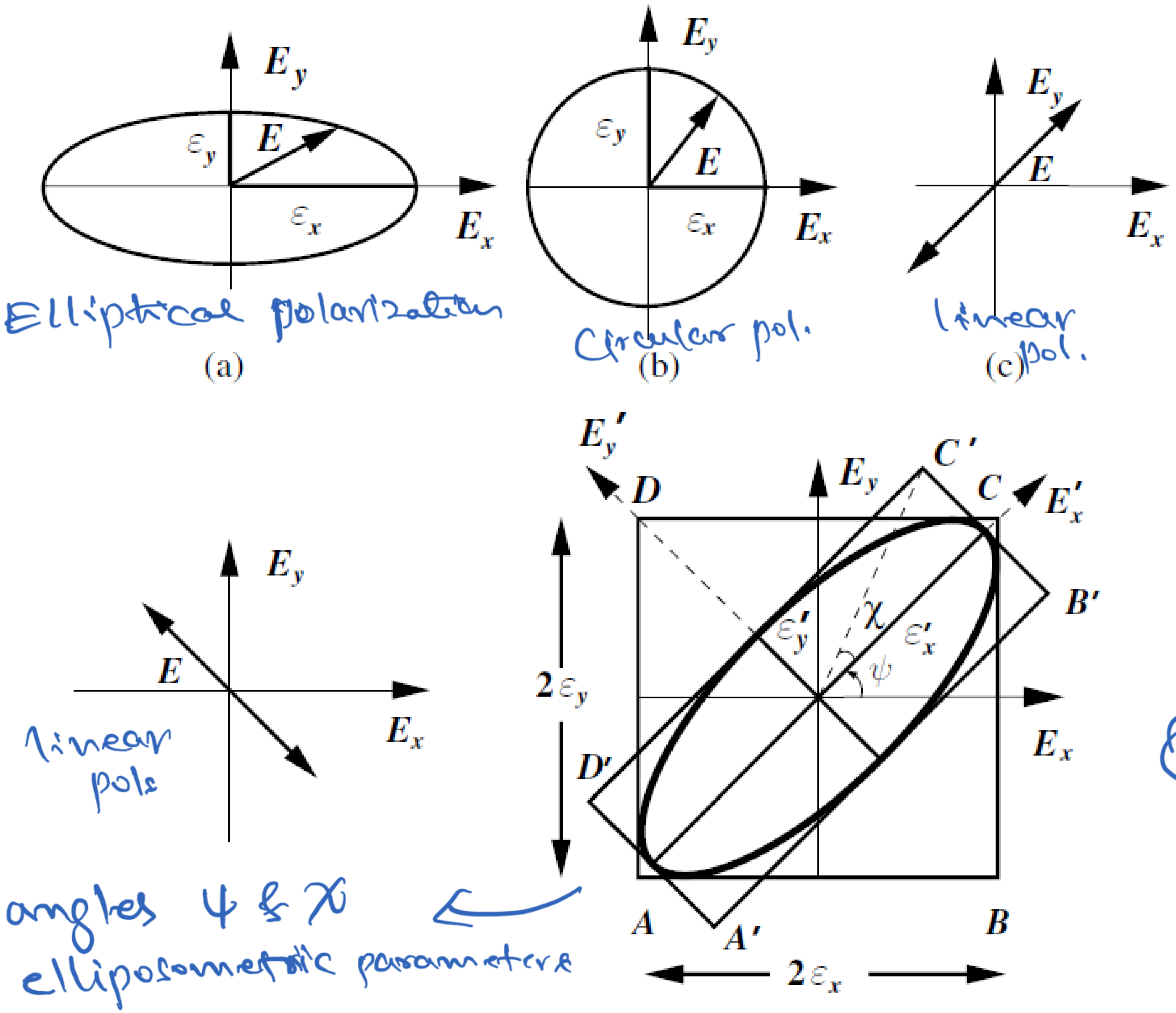
Now by using previous relation for $E_x = \mathcal{E}_x \cos(kz - \omega t)$

$$\cos^2(kz - \omega t) = \left(\frac{E_x}{\mathcal{E}_x}\right)^2$$

$$\Rightarrow \left(\frac{E_y}{\mathcal{E}_y}\right)^2 + \left(\frac{E_x}{\mathcal{E}_x}\right)^2 \cos^2\phi_0 - 2\left(\frac{E_y}{\mathcal{E}_y}\right)\left(\frac{E_x}{\mathcal{E}_x}\right)\cos\phi_0 = \left[1 - \left(\frac{E_x}{\mathcal{E}_x}\right)^2\right] \sin^2\phi_0$$

$$\Rightarrow \boxed{\left(\frac{E_y}{\mathcal{E}_y}\right)^2 + \left(\frac{E_x}{\mathcal{E}_x}\right)^2 - 2\left(\frac{E_y}{\mathcal{E}_y}\right)\left(\frac{E_x}{\mathcal{E}_x}\right)\cos\phi_0 = \sin^2\phi_0} \quad \text{--- } (3)$$

→ General equation of ellipse or polarization ellipse.



(a) (b) (c) (d) (e)

angles ψ & χ are ellipsometric parameters

(a) → For $\phi_0 = \pm(m+\frac{1}{2})\pi$

$$\Rightarrow \frac{E_y^2}{\mathcal{E}_y^2} + \frac{E_x^2}{\mathcal{E}_x^2} = 1$$

→ Elliptical polarization

(b) $\phi_0 = \pm(m+1/2)\pi$ and

$$\mathcal{E}_x = \mathcal{E}_y = \mathcal{E}$$

$$\Rightarrow E_y^2 + E_x^2 = \mathcal{E}^2$$

→ circular polarization

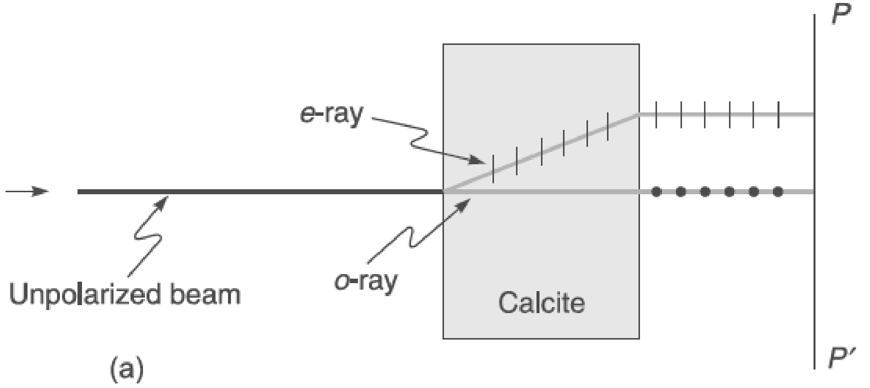
(c) $\phi_0 = 0, \pm n\pi$, $n = 1, 2, \ldots$

$$\Rightarrow E_y = \pm\frac{\mathcal{E}_y}{\mathcal{E}_x} E_x$$

→ linear polarization

(e) → General state of polarization defined by eqn (3)

DOUBLE REFRACTION IN UNIAXIAL CRYSTALS

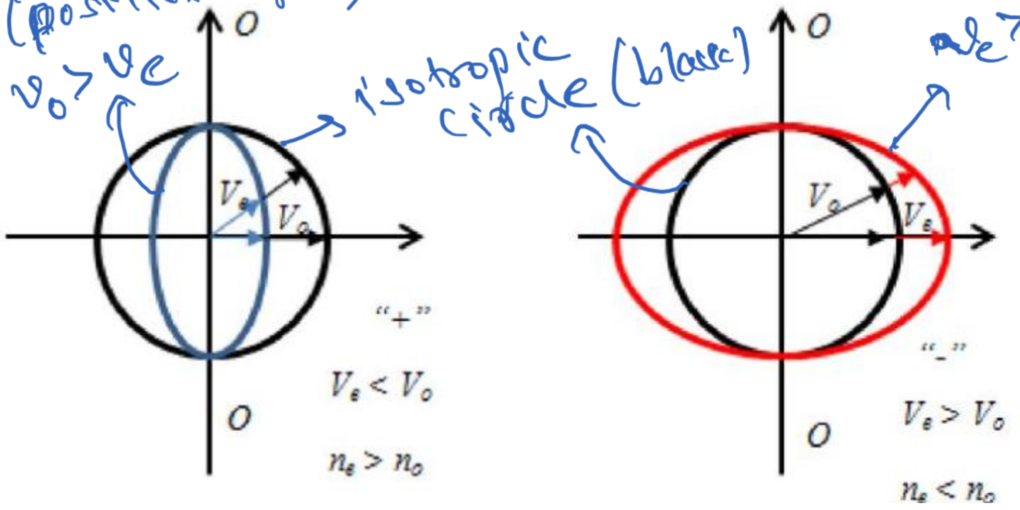


When an unpolarized light beam is incident on a calcite (uniaxial) crystal, it splits up into two linearly polarized beams along ordinary and extraordinary axes

(positive crystal) $v_o > v_e$

isotropic circle (black)

$v_e > v_o$ (negative crystal)



Positive crystal (Quartz), when $n_o < n_e$
Negative crystal (Calcite), when $n_o > n_e$

<u>Phase retarders</u>: Use of the birefringence property of anisotropic media to introduce the phase shift between the two orthogonal polarization components (E_x and E_y).

<u>Uniaxial crystals:</u> The two rays have the same velocities only along one direction (called optic axis)

The anisotropic media (e.g. Uniaxial crystals) exhibit double refraction (birefringence) property, in which the light travels through the crystal at different speeds along the orthogonal directions called Slow axis (SA) and the Fast axis (FA).

The wave polarized along the Slow axis (e.g. E_x component) moves slower inside the crystal (retarder) than the one (e.g. E_y component) polarized along the Fast axis. The propagation direction will remain same for both the waves but with different wave numbers ($k = \omega/v$).

Velocities of ordinary (n_o) and extraordinary (n_e) rays.

$$v_{ro} = \frac{c}{n_o} \qquad \text{(ordinary ray)}$$

$$\frac{1}{v_{re}^2} = \frac{\sin^2\theta}{(c/n_e)^2} + \frac{\cos^2\theta}{(c/n_o)^2} \qquad \text{(extraordinary ray)}$$

n_o and n_e are the refractive indices and θ is the angle that the ray makes with the optic axis (ordinary/slow axis)

LINEAR OPTICAL DEVICES

The difference in the velocity of the two components E_x and E_y will create phase difference of magnitude

$$\phi_0 = \phi_e - \phi_o = (k_e - k_o)d$$

$$= \frac{2\pi}{\lambda_v}(n_e - n_o)d$$

d: light propagation thickness inside the crystal,
n_o: Index of refraction for light polarized along the ordinary (slow) axis
n_e: Index of refraction for light polarized along extraordinary (fast) axis
λ_v : Wavelength of light in vacuum

For negative crystals (calcite) $(n_o > n_e) \rightarrow \phi_o \rightarrow$ Negative

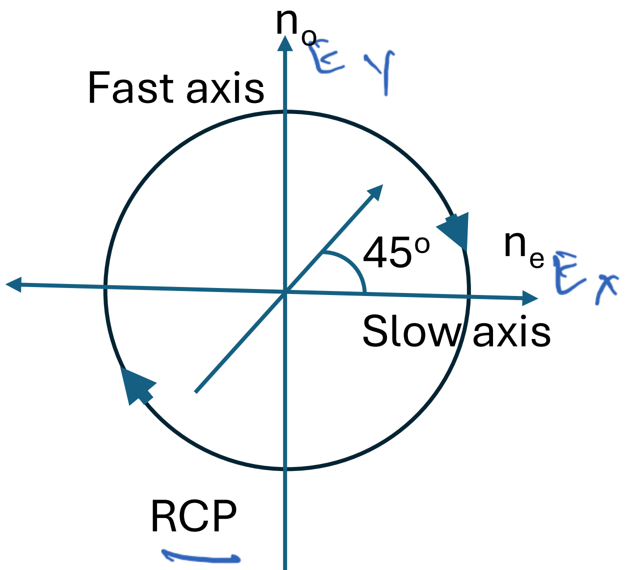
positive crystals (Quartz) $(n_o < n_e) \rightarrow \phi_o \rightarrow$ Positive

- The thickness of the phase retarder (d) must not exceed the coherence length (L_c) of the light wave.
- The phase difference (φ_0) introduced by the phase retarder transforms a plane polarized wave into an elliptically polarized wave with circular and linear polarizations as special cases depending on the thickness d of the phase retarder and angle θ between the polarization direction of the incident light and the <u>slow axis</u> of the phase retarder.
- The phase retarder changes only the phase, and not the amplitude of the wave.

LINEAR OPTICAL DEVICES

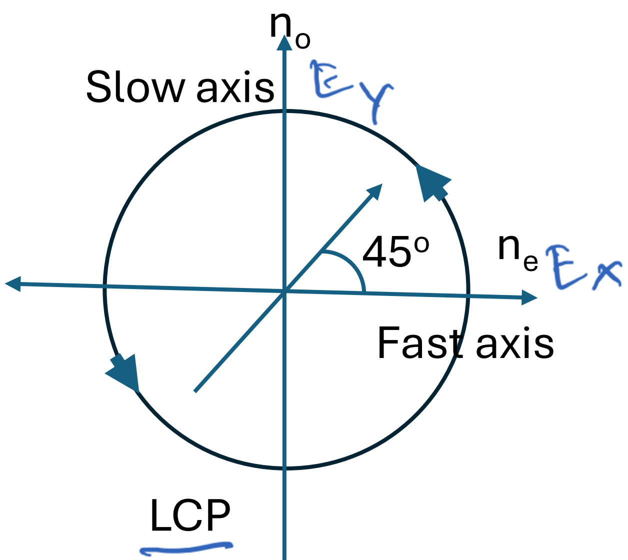
- The Right circular polarization (RCP) or Left Circular Polarization (LCP) is defined by the rotation made of the polarized light (at 45°) towards the slow (ordinary) axis of the crystal

- If the **slow** (ordinary) axis is in the 'horizontal' direction, then the polarized light at 45° will be converted to Right circular polarization (RCP): The phase difference (φ_o) between Slow and Fast axes is negative.
- If the **slow** (ordinary) axis is in the 'vertical' direction, then the polarized light at 45° will be converted to Left circular polarization (LCP): The phase difference (φ_o) between Slow and Fast axes is positive.



Quartz ($n_o < n_e$)
φ_o is Positive

$v_{Ex} < v_{Ey}$

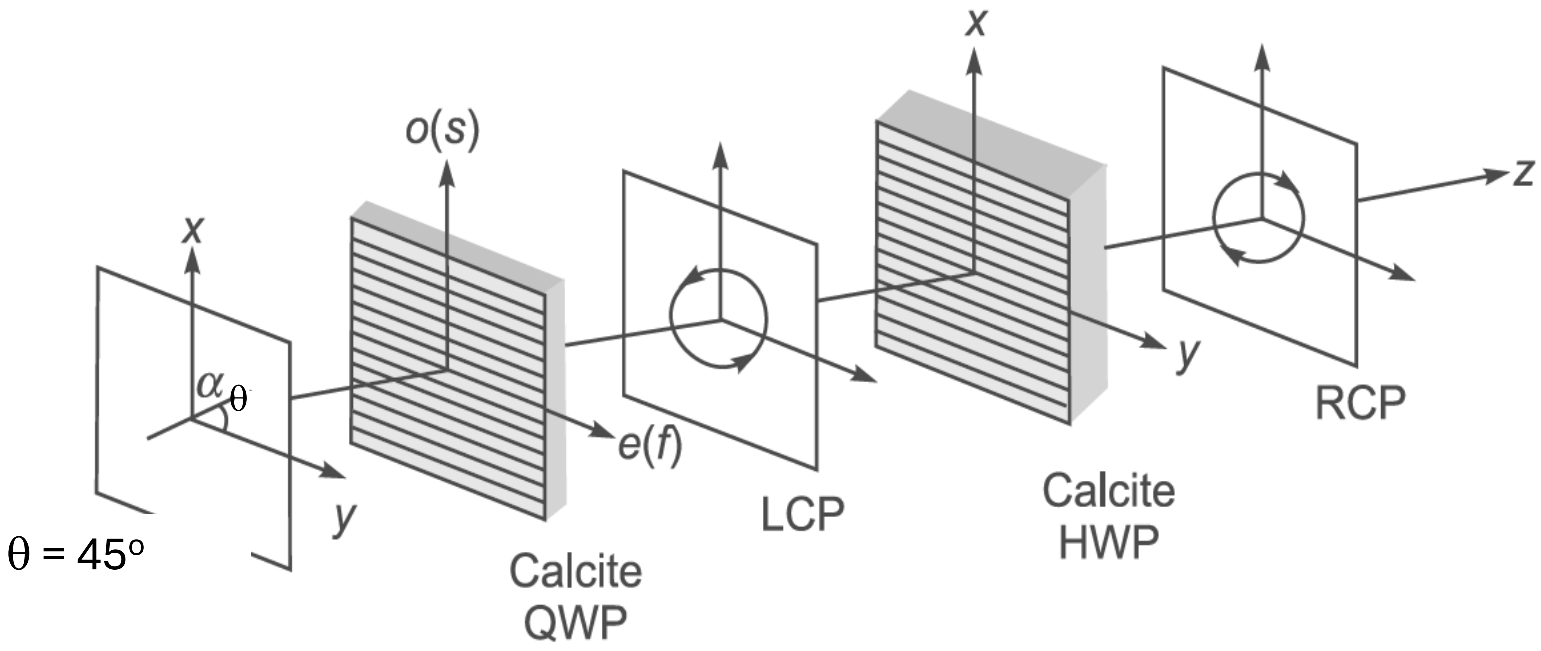


Calcite ($n_o > n_e$)
φ_o is negative

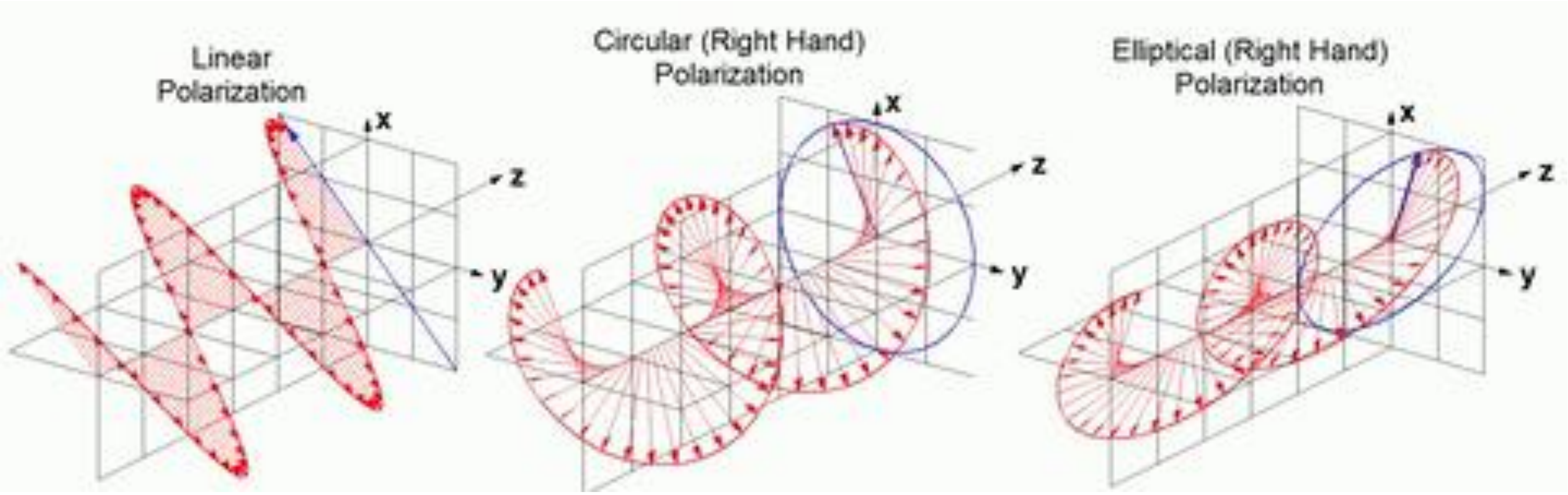
$v_{Ex} > v_{Ey}$

E_x lags E_y for RCP
→ Positive phase difference (ϕ_o)

E_x leads E_y for LCP
→ Negative phase difference (ϕ_o)

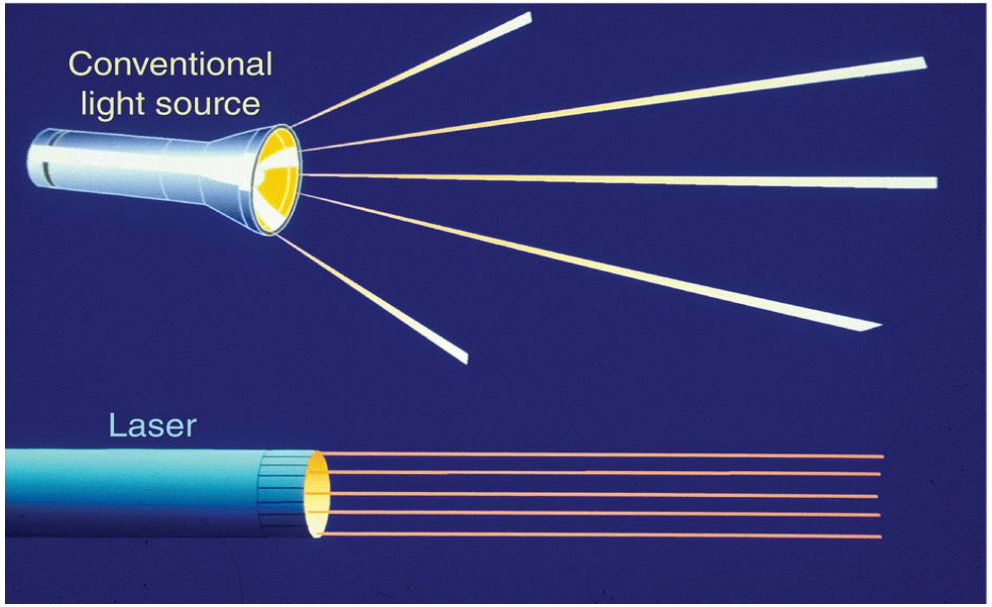


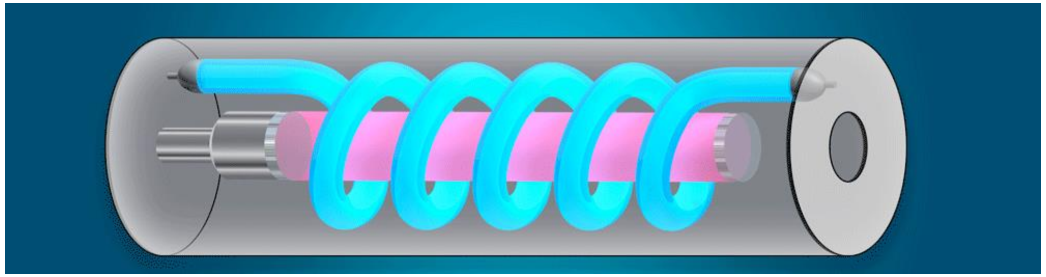
x
o(s)
x
z
α
θ
e(f)
y
y
y
θ = 45°
Calcite QWP
LCP
Calcite HWP
RCP



Linear Polarization
Circular (Right Hand) Polarization
Elliptical (Right Hand) Polarization
x
z
y
x
z
y
x
z
y

LASER: Light Amplification by Stimulated Emission of Radiation





Townes, Basov and Prochorov were awarded the **1964 Nobel Prize in physics** for their fundamental work in the field of Quantum Electronics and for MASER-LASER Discovery.

Special characteristics of LASERs

- Directionality: very low divergence ($\Delta\theta < 10^{-5}$ radians) → long distance propagation
- High Intensity: very high intensity → nonlinear interactions ($I = \frac{\text{power}}{\text{area}}$)
- Spectral purity ($\Delta\lambda$): Monochromatic (nearly single wavelength) → extremely small spectral width.
- Tight Focusing: single wavelength → very tight focusing (diffraction limited)
- Coherent: constant phase & amplitude → interferometers (sensitive)
  coherence length $L_c = \lambda^2/\Delta\lambda$ ~ meters/kms.

MAIN COMPONENTS OF THE LASER

→ Laser is a source of EM waves

→ As a source, it is analogous to an 'Electronic oscillator' (a source of RF)

components of electronic oscillator → power supply (Vin)

→ Active component (Op. Amp) } Amplified

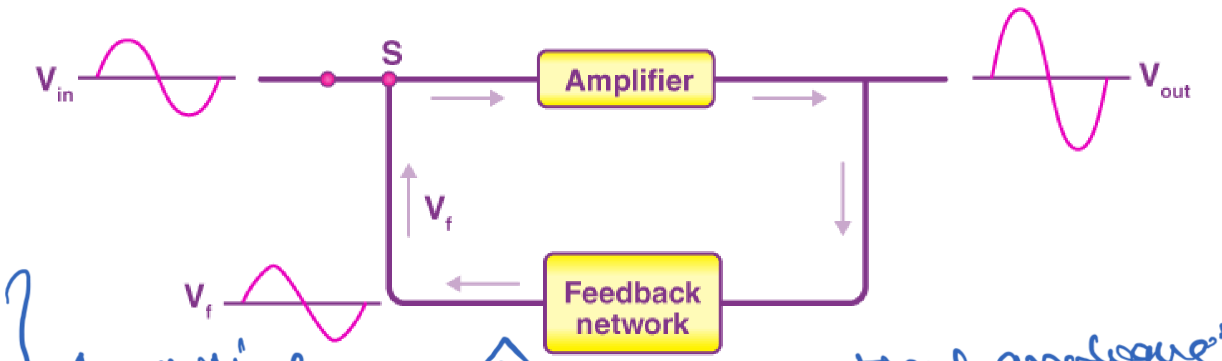
→ Feedback circuit (phase stability) (RC circuit)

Amplifier + Feed back circuit → Oscillator

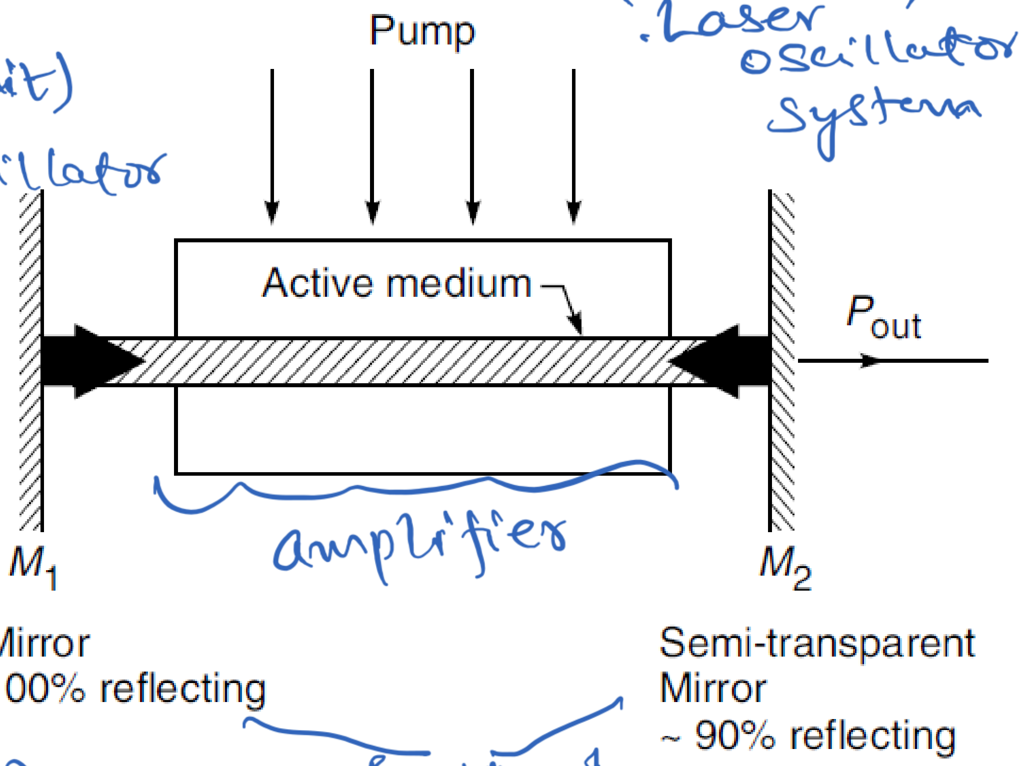
→ Active medium provides amplification when pumped suitably.

→ Optical resonator → The mirrors provide optical feedback. (a few photons that remain in the cavity drive the atoms coherently to create the lasing action)

Electronic oscillator circuit



optical analogue: Laser oscillator system

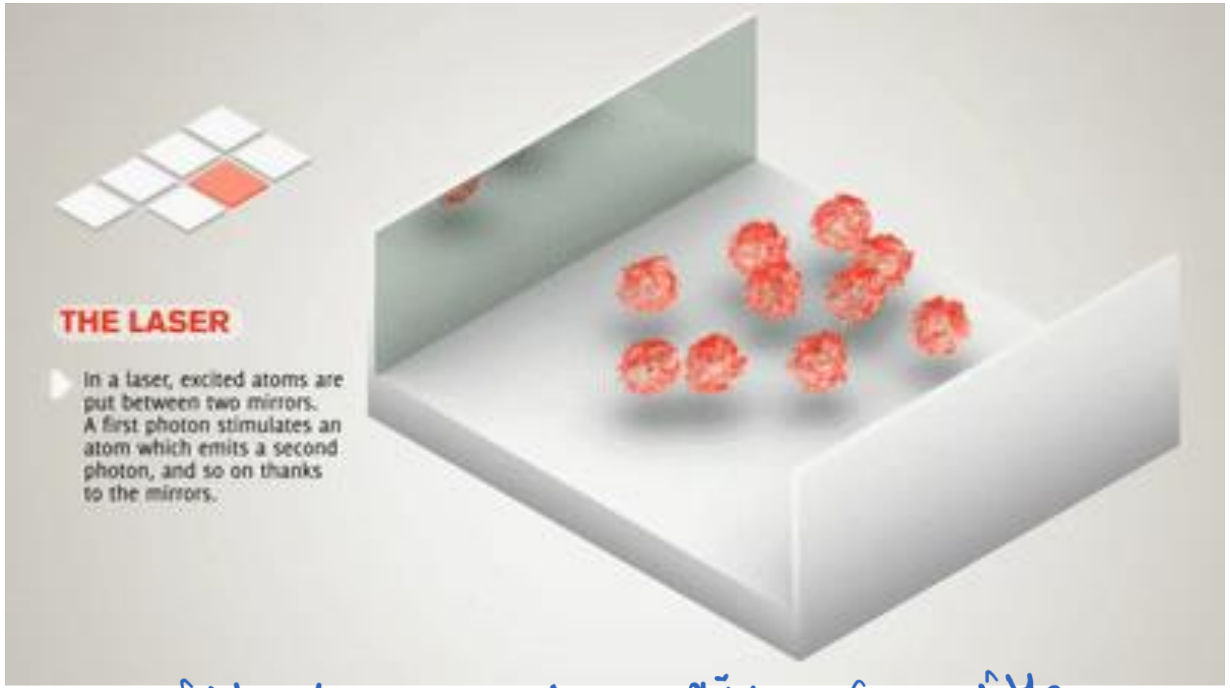


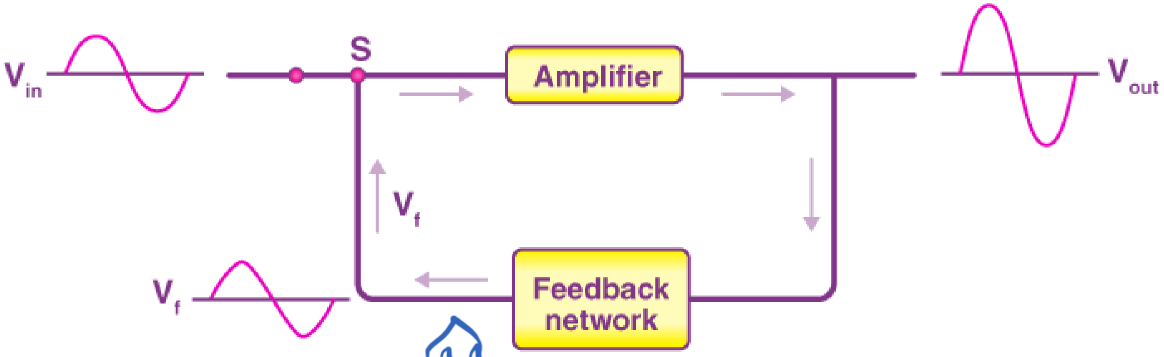
amplifier

feedback cavity.

MAIN COMPONENTS OF THE LASER

Electronic oscillator circuit (with feedback)



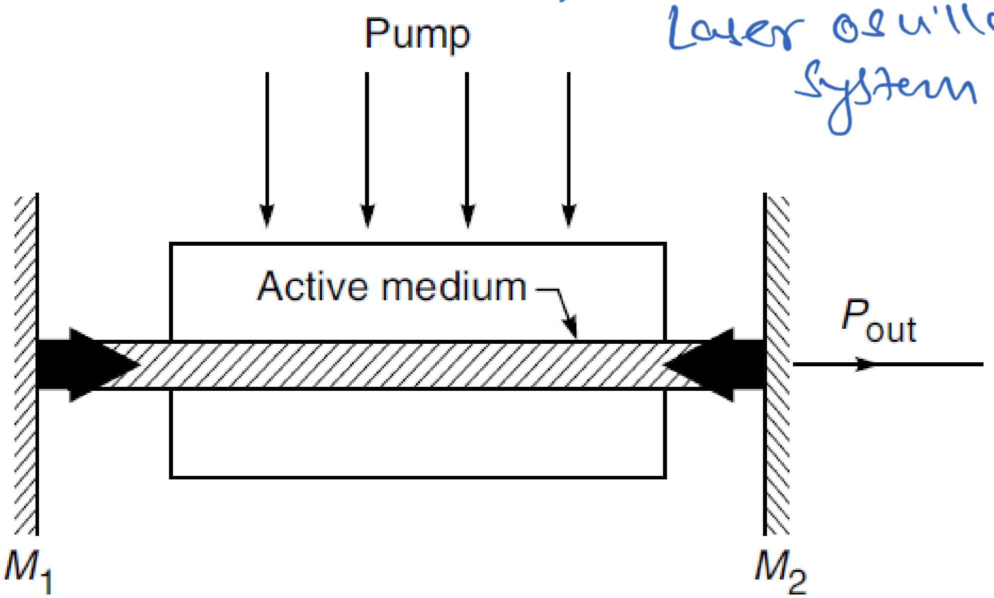


→ optical analogue Laser oscillator system

oscillator = Amplifier with feed back

Amplifier
- ✓ The Active medium → Gain medium (atoms, molecules, solid state materials)
- ✓ The Pumping source → optical field with high energy / intensity
- ✓ The Optical Resonator → optical cavity with high Quality factor (low photon losses)

(feed back)



Mirror
100% reflecting

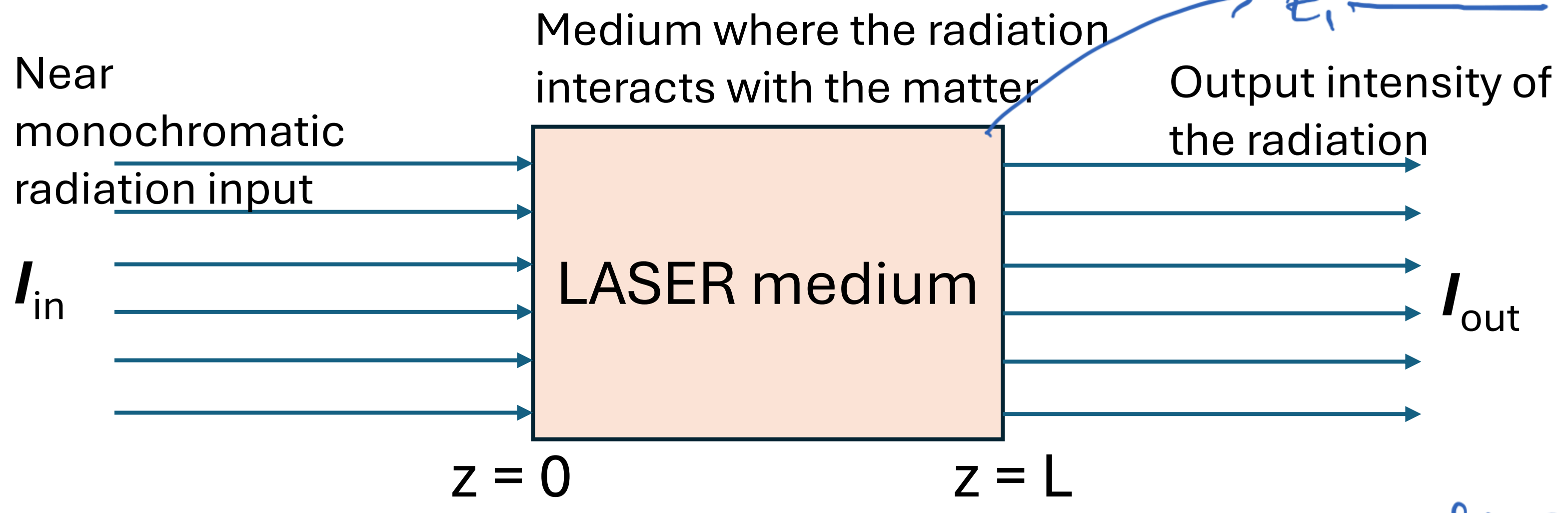
Semi-transparent
Mirror
~ 90% reflecting

OPTICAL GAIN AND POPULATION INVERSION

E_2 —— N_2 → population in the upper energy state

$I_{in} \to h\nu$

E_1 —— N_1 → population in the lower energy state



The output intensity of light after propagating the medium of length L, is

$$I_{out} = I_{in} e^{\gamma L}$$

Gain coefficient → $\gamma \propto \frac{(N_2 - N_1)}{t_{sp}}$

N_1 & N_2 are population in E_1 & E_2 energy states

t_{sp} → spontaneous life time

if $\gamma < 0 \Rightarrow I_{out} < I_{in}$ → Loss in the medium (absorption)

$\gamma > 0 \Rightarrow I_{out} > I_{in}$ → Gain in the medium

OPTICAL GAIN AND POPULATION INVERSION

For $\gamma > 0$, $N_2 - N_1 > 0 \Rightarrow N_2 > N_1$

i.e the population in the upper energy state (E_2) should be greater than the population in the lower energy state (E_1)

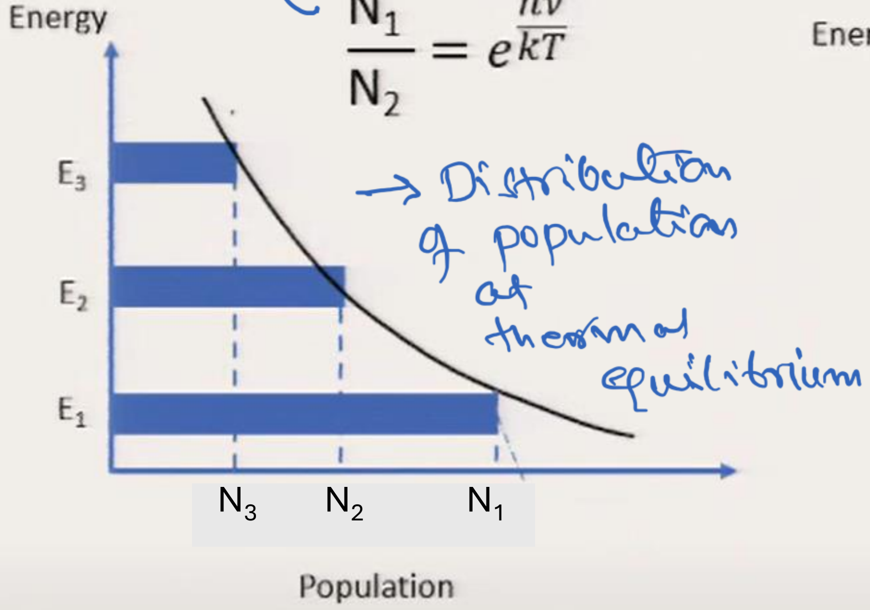
$\Rightarrow$ population inversion → This is the necessary condition for Amplification by Stimulated emission.

↓

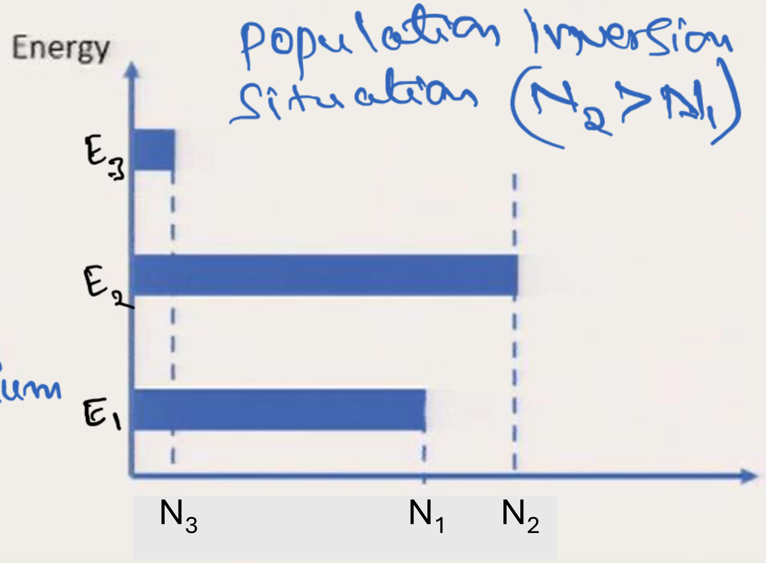
leads to the stimulated emission and lasing action with feed back from the cavity (optical resonator)

Boltzmann distribution

$$\frac{N_1}{N_2} = e^{\frac{h\nu}{kT}}$$

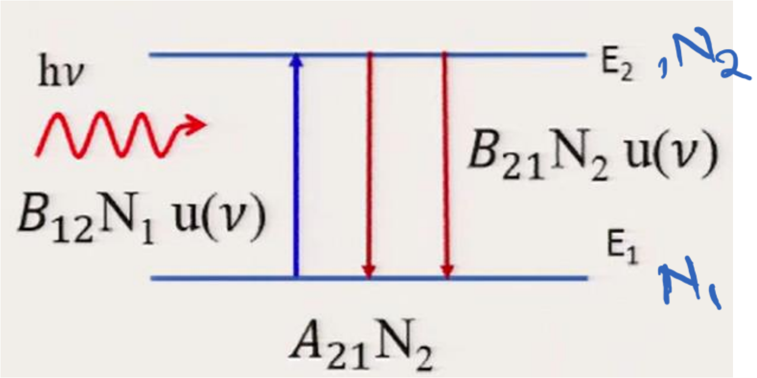


Population diagram in thermal equilibrium.



Population Inversion in the presence of an external pump.

2-LEVEL ATOMIC SYSTEM



At thermal equilibrium,

Number of upward transitions = Number of downward transitions

$$A_{21}\,N_2 + B_{21}N_2\,u(\nu) = B_{12}\,N_1\,u(\nu)$$

⇒ Rate of emission = Rate of absorption

⇒ There is always an equal destribution of number of atoms in the ground (E_1) state and the excited (E_2) state

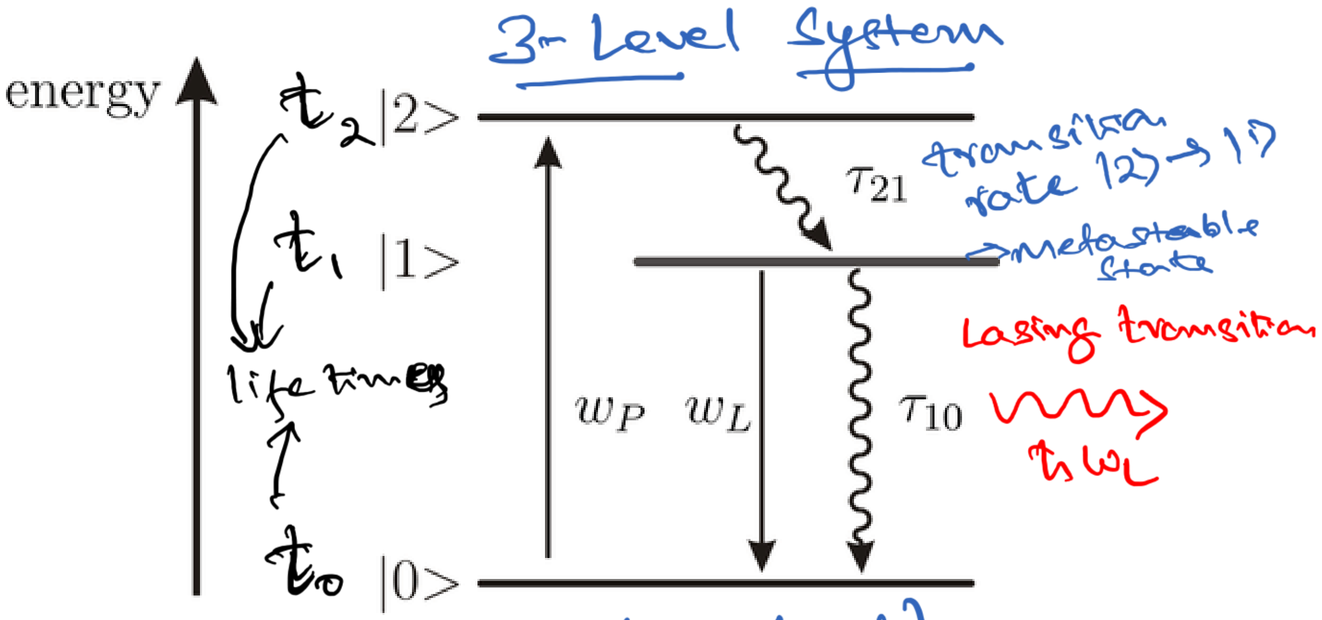
∴ $N_2 \le N_1$ at thermal equilibrium (Steady state)

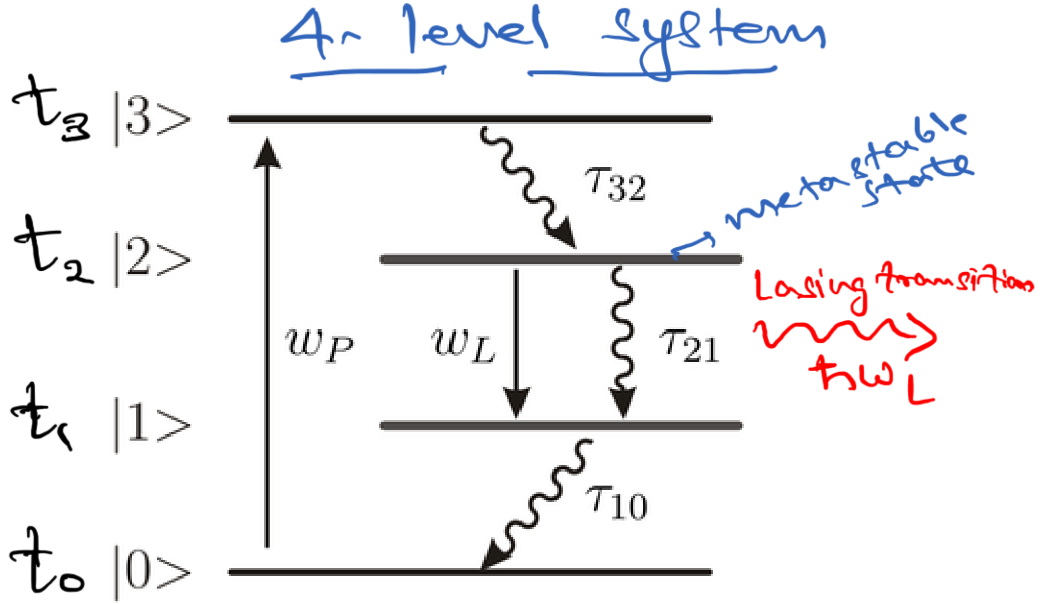
⇒ No-population inversion possible for two-level energy system in the steady state ⇒ Not suitable for Laser medium

N_1 → Number of population per unit time per unit volume in energy E_1 state (ground state)

N_2 → No. of population per unit time per unit volume in energy E_2 state (excited state)

3-LEVEL AND 4-LEVEL ATOMIC SYSTEM: LASING REQUIREMENTS





$|0\rangle \rightarrow$ Ground state (long lived)

$|1\rangle \rightarrow$ Intermediate (metastable state) $\rightarrow$ supports population inversion

$|2\rangle \rightarrow$ Excited state (short lived)

$|0\rangle \rightarrow |2\rangle$: Pumping transition

$|1\rangle \rightarrow |0\rangle$: Lasing transition

$|0\rangle \rightarrow |3\rangle \rightarrow$ pumping transition

$|2\rangle \rightarrow |1\rangle \rightarrow$ lasing transition

$\rightarrow$ Ground state is not involved in the lasing transition, it is easier get population inversion in 4-level system

Lifetimes: $\tau_{00} > \tau_{10} > \tau_{21}$

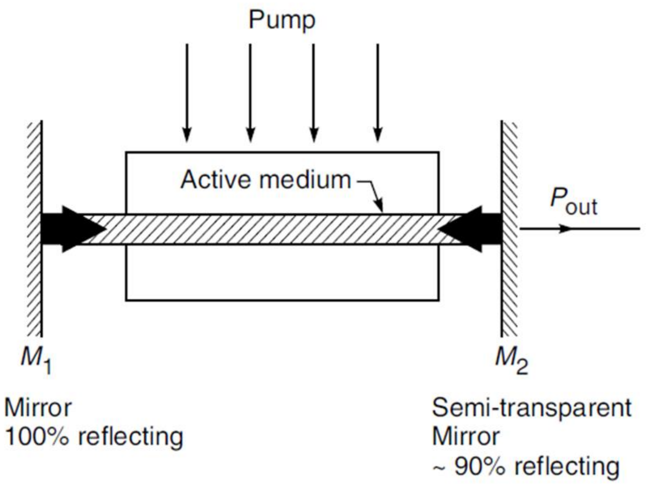
Lifetimes: $\tau_{00} > \tau_{21} > \tau_{10} > \tau_{32}$

3-LEVEL AND 4-LEVEL ATOMIC SYSTEM: LASING REQUIREMENTS

3-Level Laser system	**4-Level Laser system**
Requires 3-energy non-degenerate energy levels	Requires 4-energy non-degenerate energy levels
The terminal level is the Ground state and hence it requires more than half of atoms (>50%) are to be transferred to the metastable state for the population inversion condition	Since the Ground state is not involved in the Laser transition states, therefore a any number of atoms greater than the energy level |1> can give rise to Population inversion
Can realize population inversion	More-easier to realize Population inversion
Optimum/less Laser Efficiency (more number of spontaneous emission)	Much better Laser emission efficiency (Less spontaneous emission)
This requires more pumping power (higher pumping rates)	Requires less pumping power (Lower pumping rates)
Only Pulsed Laser Operation is Possible	Both Continuous and Pulsed Laser operation is possible
Higher Lasing Threshold	Lower Lasing Threshold
E.g. Ruby laser	E.g. He-Ne laser, Nd-YAG Laser

4-Level Laser system is Better than the 3-Level Laser system

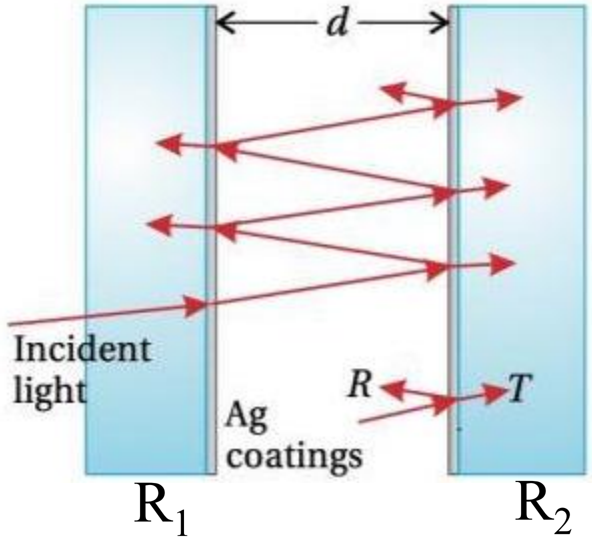
OPTICAL FEEDBACK: OPTICAL CAVITY/RESONATOR

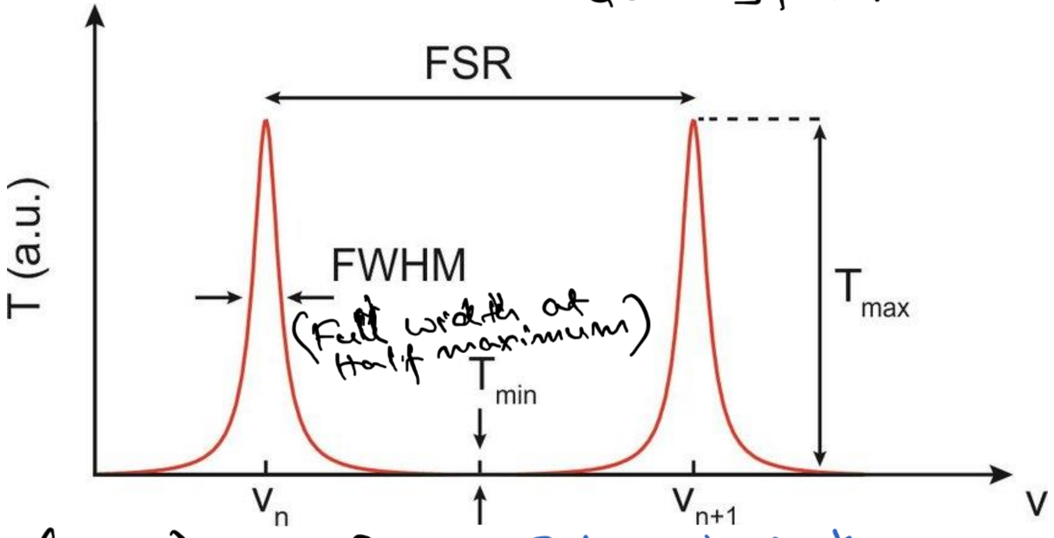


- A system of two parallel mirrors reflecting (confining) the light within : Resonators/Cavities
- A laser cavity/resonator acts a feedback to provide a portion of energy back into the active (amplified) medium to create a sustained stimulated emission for coherent laser action.

Fabry-Perot resonator/interferometer

Cavity Finesse or Quality factor, $F = \frac{\pi \sqrt[4]{R_1 R_2}}{1 - \sqrt{R_1 R_2}} = \frac{FSR}{FWHM}$ } → Quantifies the losses in the cavity





cavity resonance

$$\nu_n = n \frac{c}{2 n_0 d}$$

n → order of modes $n = 1, 2, 3, \ldots$

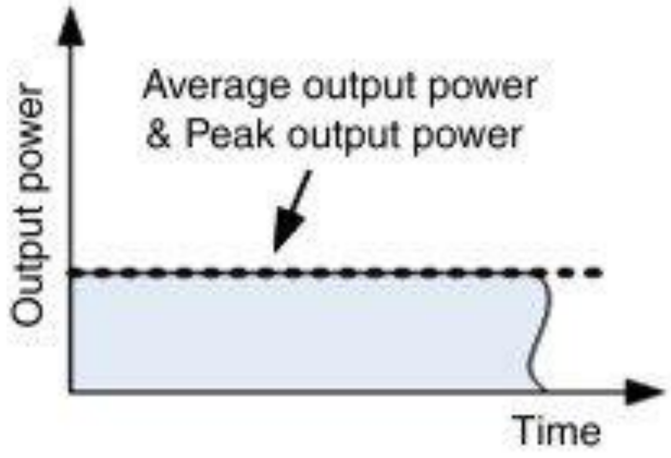
n_0 → Refractive index of the medium inside cavity

d → spacing/distance b/w the two mirrors.

Free spectral Range (FSR) $(\nu_{n+1} - \nu_n)$ → $= \frac{c}{2 n_0 d}$ → Determines the number of modes in a laser cavity.

TYPES OF LASERS

- **Continuous Lasers**: The lasers wherein the output of the laser is characterized by continuously distributed power in time.

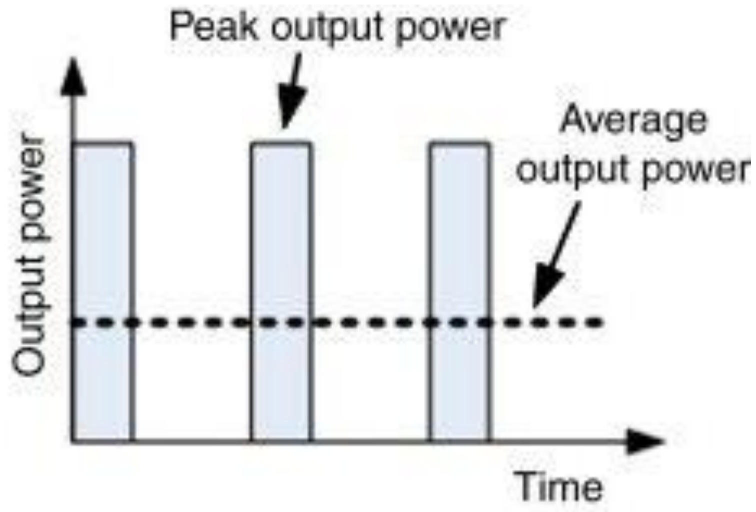


(a) Continuous wave (CW) laser beam

→ The reduced re-absorption of stimulated photons in the 4-level system gives continuous Laser operation

→ peak output power is same as average output power

→ Low energy density or intensity



(b) Pulsed laser beam

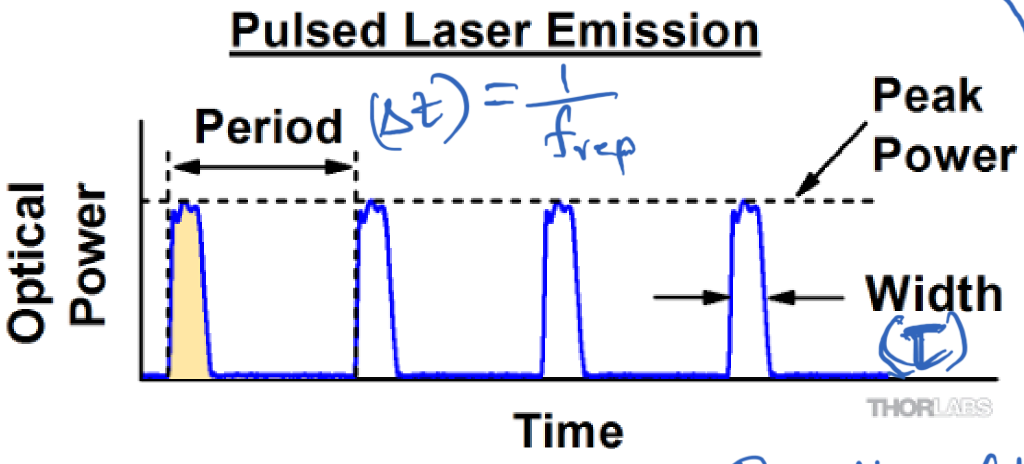
Laser output is desmitized with peak output power for a given pulse is much higher than the average power

- Monochromatic laser
- 4-Level atomic lasers (He-Ne (632nm), Nd-YAG lasers (1064 nm))
- Diode/Semiconductor Lasers (Tunable)

TYPES OF LASERS

- **Pulsed Lasers**: The lasers wherein the output of the laser comprise of the light pulses that are periodic in time

Parameters that characterise pulsed laser →



Energy $E = \frac{P_{avg}}{f_{rep}} = P_{avg} \cdot \Delta t$ (J)

avg power $P_{avg} = \frac{E}{\Delta t} = E \cdot f_{rep}$ (W)

peak power $P_{peak} = \frac{P_{avg}}{f_{rep} \cdot \tau} = \frac{P_{avg} \cdot \Delta t}{\tau}$ (W)

Δt	Pulse Period
E	Energy per Pulse
f_{rep}	Repetition Rate
P_{avg}	Average Power
P_{peak}	Peak Power
τ	Pulse Width

Realizable → 3-level energy system
→ By modulating the continous laser output (External, Q-switching, mode locking, ...)

Applications:

1. **Optical Communication**: Binary Digital communication with ON and OFF pulses at very high data rates.
2. **Laser Surgery/Precision Cutting**: Highly localized ablation of materials/tissues.
3. **Nonlinear Optical Processes**: High Peak Powers can aid in inducing the nonlinear optical processes in mat
4. **Ultrafast Processes**: Investigate the processes in the time scales of ~ 10^{-9} (nano-sec) – 10^{-18} (atto-sec)

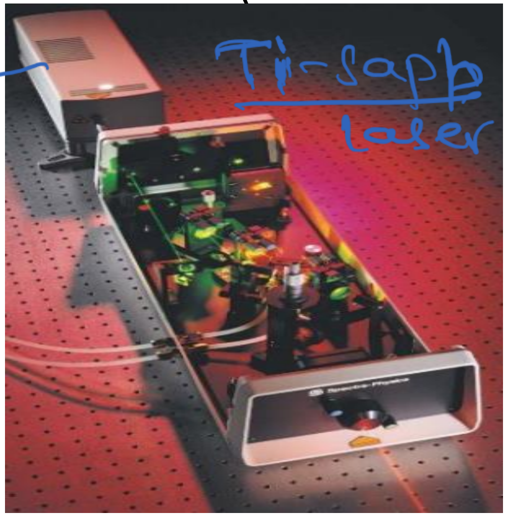
TYPES OF LASERS

Classification based on the type of active (gain) medium or pumping scheme employed

1. **Gas Lasers**: He-Ne (632nm), Ar-ion (~500nm), N_2 (337nm), CO_2 (10μm) lasers

   Pumped using electric discharge mechanism

2. **Solid-state Lasers:** Ruby (694nm), Nd:YAG (~1064 nm), Ti-Sapp (670-1100nm), Fiber Lasers (1030nm – 2050 nm)



Pump (Green laser)

Ti-Sapp laser

   Pumped using a flash lamps or by diode lasers or by electrical signals

3. **Liquid Lasers:** Dye Laser, Rhodamine 6G (Visible light)

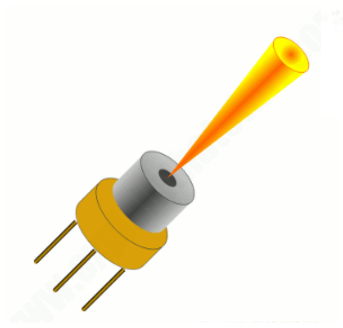
   Pumped using UV sources (SHG)

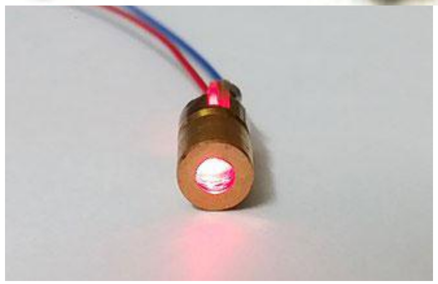
4. **Semiconductor Lasers**: Heterostructure p-n diodes (AlGaAs/GaAs, InGaAsP/InP) (Tunable wavelength); Quantum Cascade Lasers, QCLs (3um (Mid-IR) – 150um (THz))

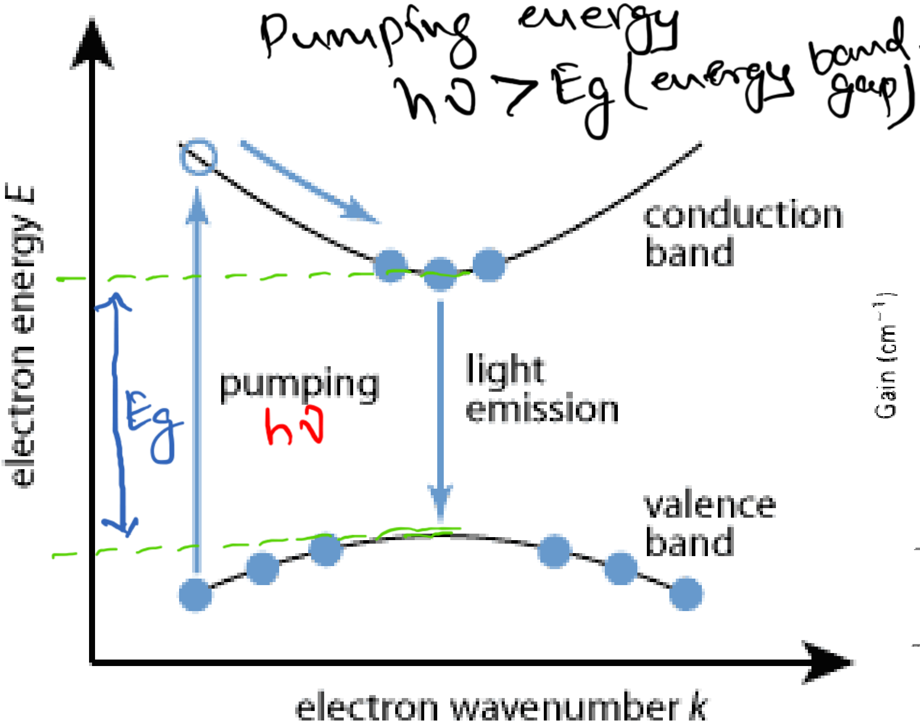
   Pumped by Injection current through the forward biased p-n junction

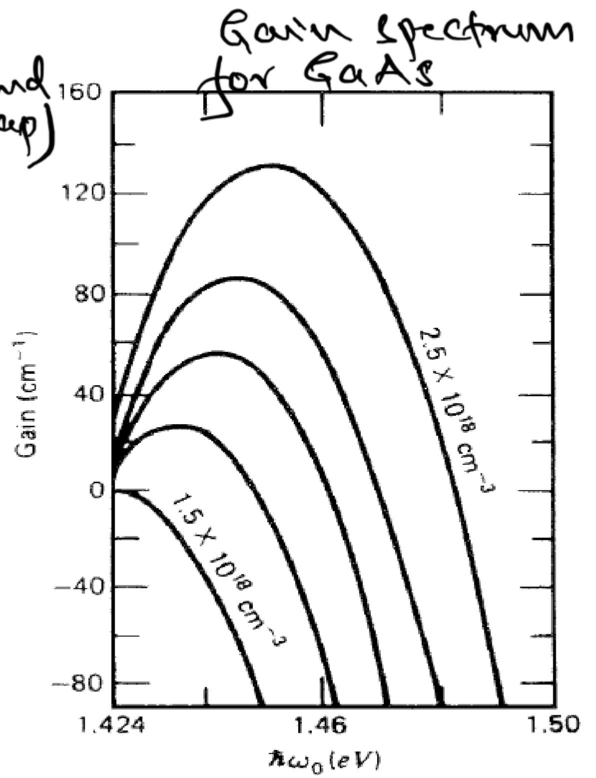
SEMICONDUCTOR LASER

Invented in 1962









Advantages

- Compact ~ µm - mm
- Efficient ~ $\eta = \frac{P_{out}\text{(optical)}}{P_{in}\text{(electrical)}}$ ~ 30% - 60%
- Direct Modulation (Electrical) ~ Electrical pulse (Gain Switching)
- Optoelectronic integration (On-Chip) → CMOS compatible
- Tunable wavelength (Spectrum) → wide gain spectrum
- Cost-effective → relately cheap price than other lasers
- Single mode and Multimode operation → $\Delta\nu$ ~ kHz (very narrow line width)
- Continuous Laser ~ power ~ mW

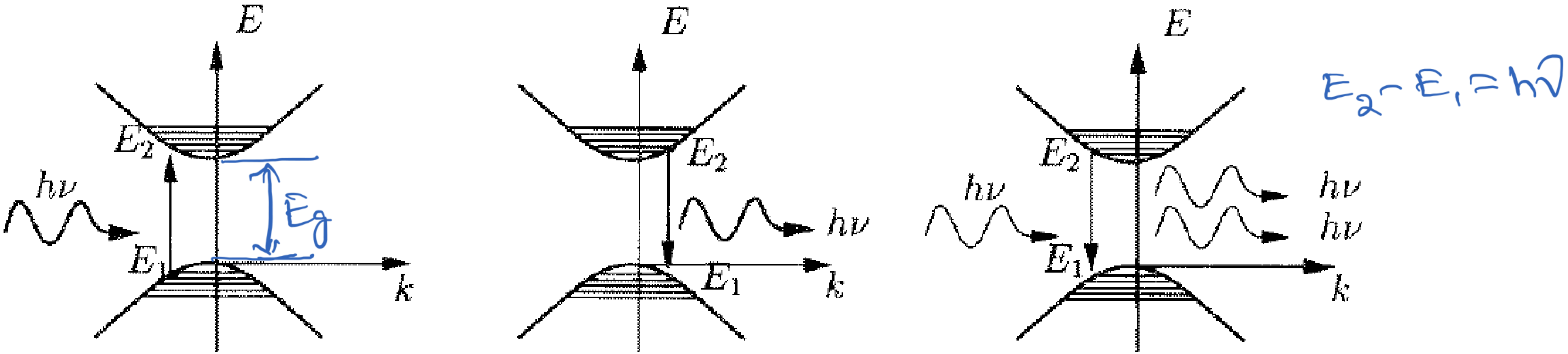
→ Energy levels
→ semiconductor bands (conduction band valence band)

→ Direct band-gap materials

→ GaAs, InP, InGaAs AlGaAs ...

→ electron density dependent gain spectrum.

SEMICONDUCTOR LASER



Absorption ($h\nu \geq E_g$)

Spontaneous emission

Stimulated emission

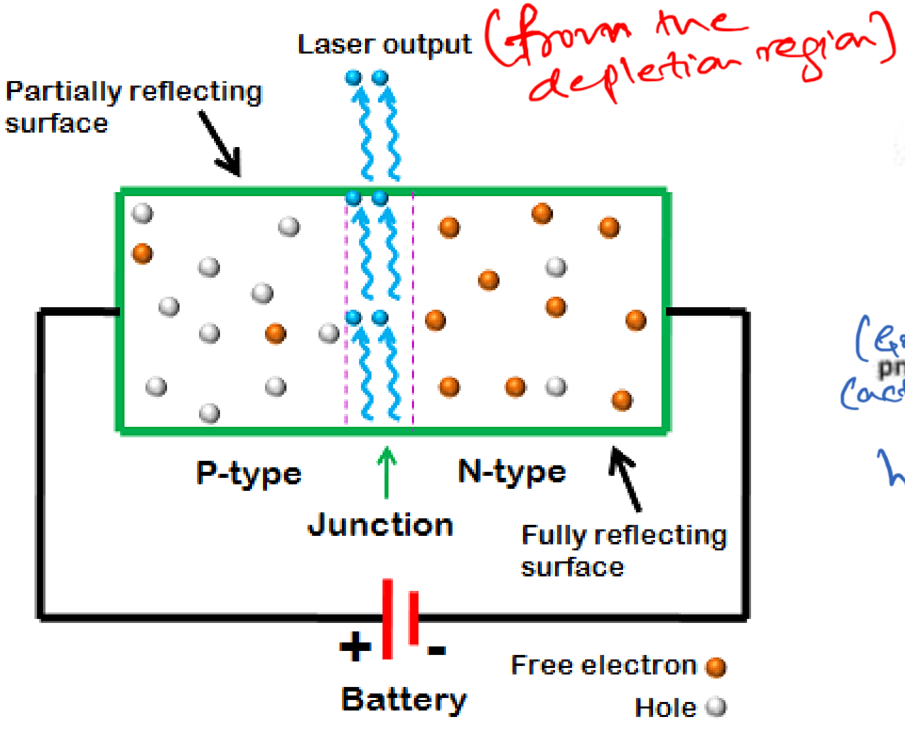
→ The stimulated emission can only happen in the direct bandgap semiconductors, where $\Delta k = k_2 - k_1 = 0$

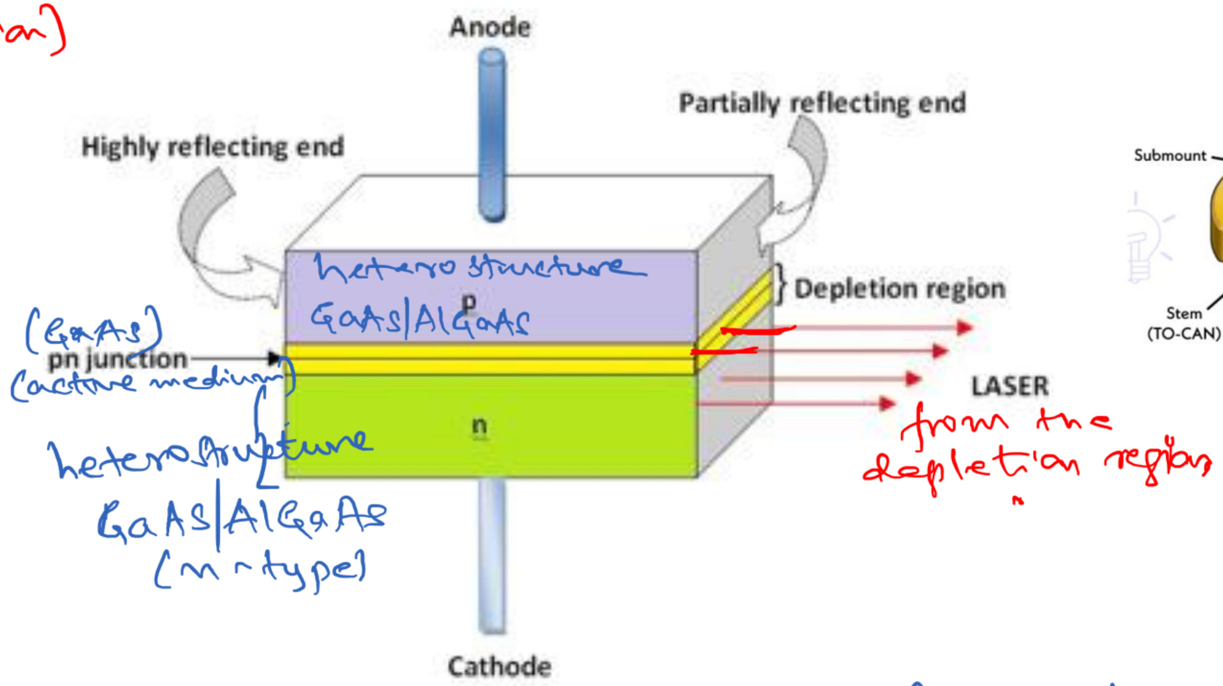
⇒ energy and the momentum of the participating photons are conserved.

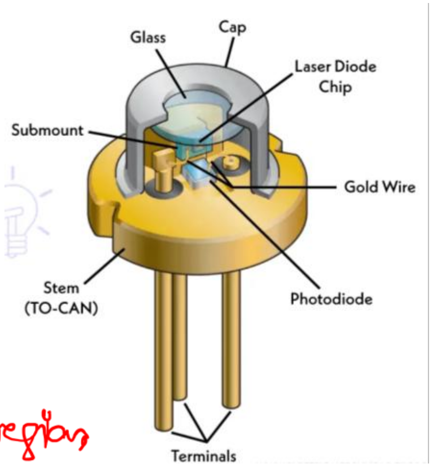
→ In the in-direct bandgap transitions (emissions), the momentum cannot be conserved → Therefore the indirect bandgap materials do not emit any stimulated photons.

SEMICONDUCTOR LASER

Bandgap of GaAs → 1.42 eV (870 nm)
AlGaAs → 1.42 eV to 2 eV







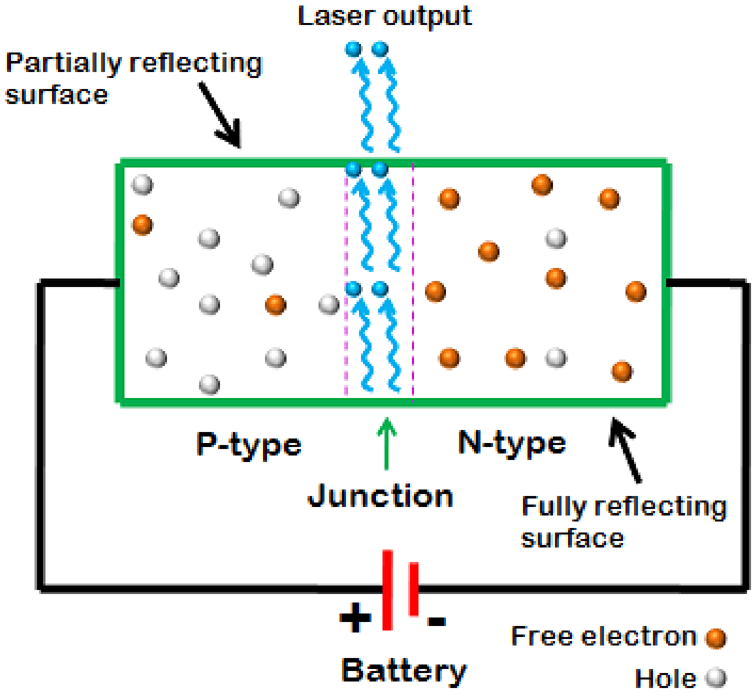
→ A p-n junction device in Forward bias made up of a direct bandgap semiconductor materials (eg. GaAs, InP, ...)

→ The junction forms a depletion region, which composed of high refractive index material that acts as an active medium.

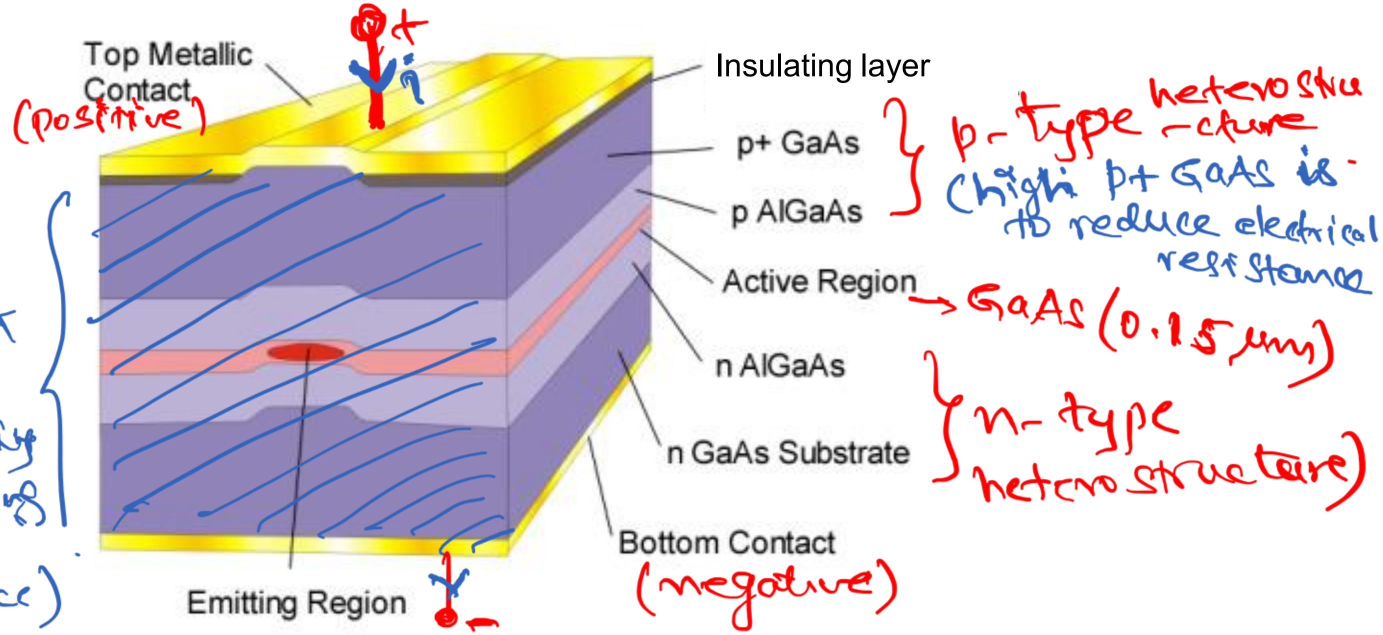
→ The Forward bias will create a electron-hole recombination at the depletion region, that makes the basis for light emission

→ The side walls of the junctions are cleaved so that entire structure acts as a Fabry-perot resonator, that provides the feedback for lasing.

SEMICONDUCTOR LASER



Double Hetero structure diode laser device configurations



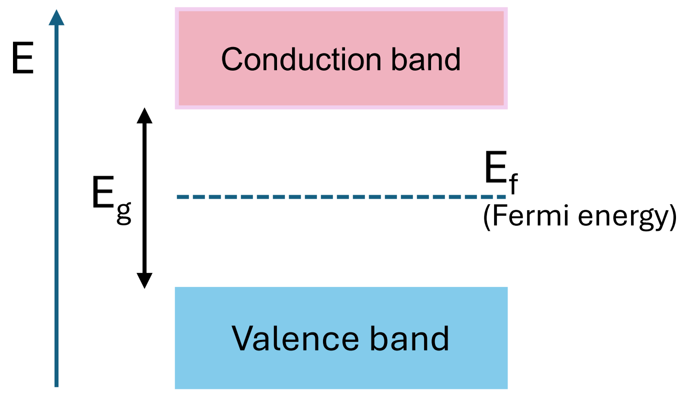
→ The laser diode configuration is a double heterostructure p-GaAs / p-AlGaAs / Junction / n-AlGaAs / n-GaAs

$n_{GaAs} = 3.6$

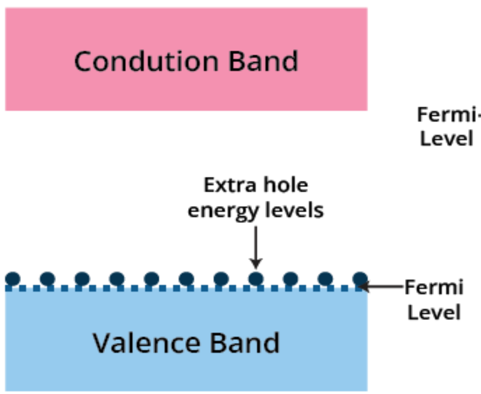
$n_{AlGaAs} = 3.4$

→ Advantages :- → strong carrier confinement
→ high optical confinement (because of different index of refraction (n) for AlGaAs & GaAs → Total internal reflection
→ lower absorption losses
→ Design flexibility.

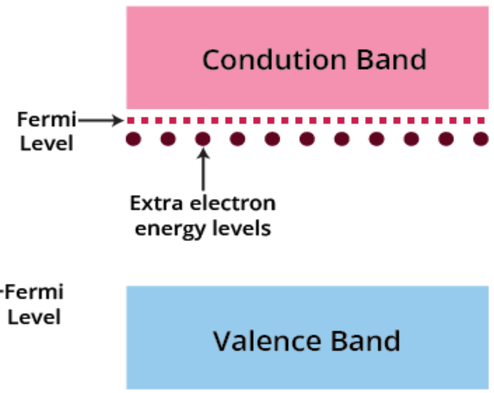
ENERGY BAND DIAGRAM OF P-N JUNCTION LASER DIODE



Intrinsic Semiconductor



P - Type

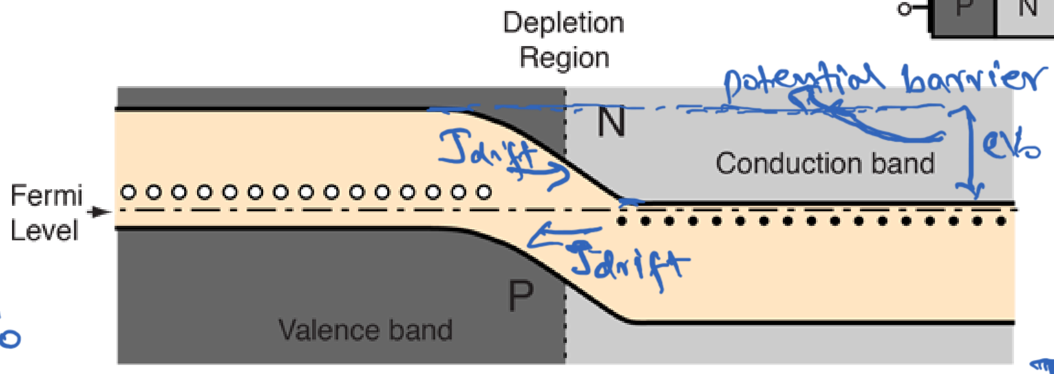


N - Type

P-type → Fermi level stays near the valence band (acceptor energy level)

n-type → Fermi level is near conduction band (donor energy level) (more electrons)

After P-n contact → forms the 'junction' (diffusion & drift of charges)
potential barrier ~ eV_0

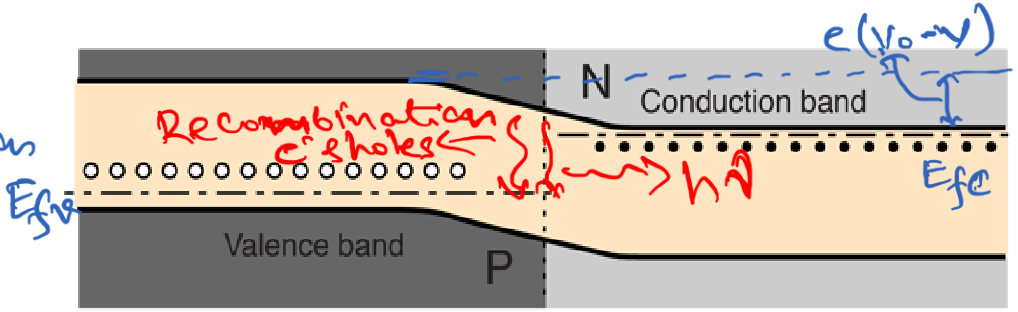


p-n junction

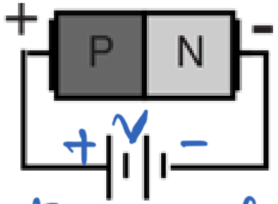
At equilibrium (no-bias) → The fermi level is alinged at the junction
$\Rightarrow J_{drift} + J_{diff} = 0$

J → current density at the junction

<u>Forward bias</u>
→ Reduced potential barrier at the junction ~ $e(V_0 - V)$
V → applied voltage



p-n junction

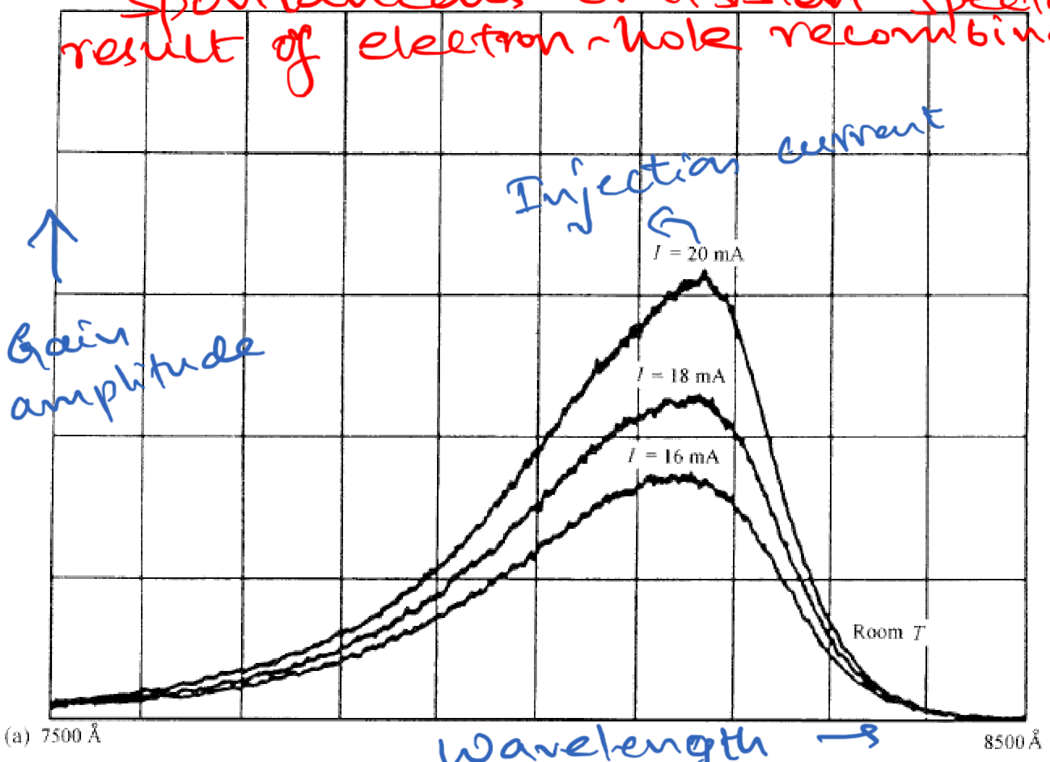


Forward bias

E_{fv} & E_{fc} are quasi-Fermi levels

SEMICONDUCTOR LASER: OPTICAL GAIN SPECTRUM

spontaneous emission spectrum
result of electron-hole recombination



Gain co-efficient for stimulated emission is,

$$\gamma(\nu) = \frac{(\frac{c}{n})^2}{8\pi\tau_r\nu^2}\frac{(2m_r)^{3/2}}{\pi\hbar^2}(h\nu - E_g)^{1/2}$$

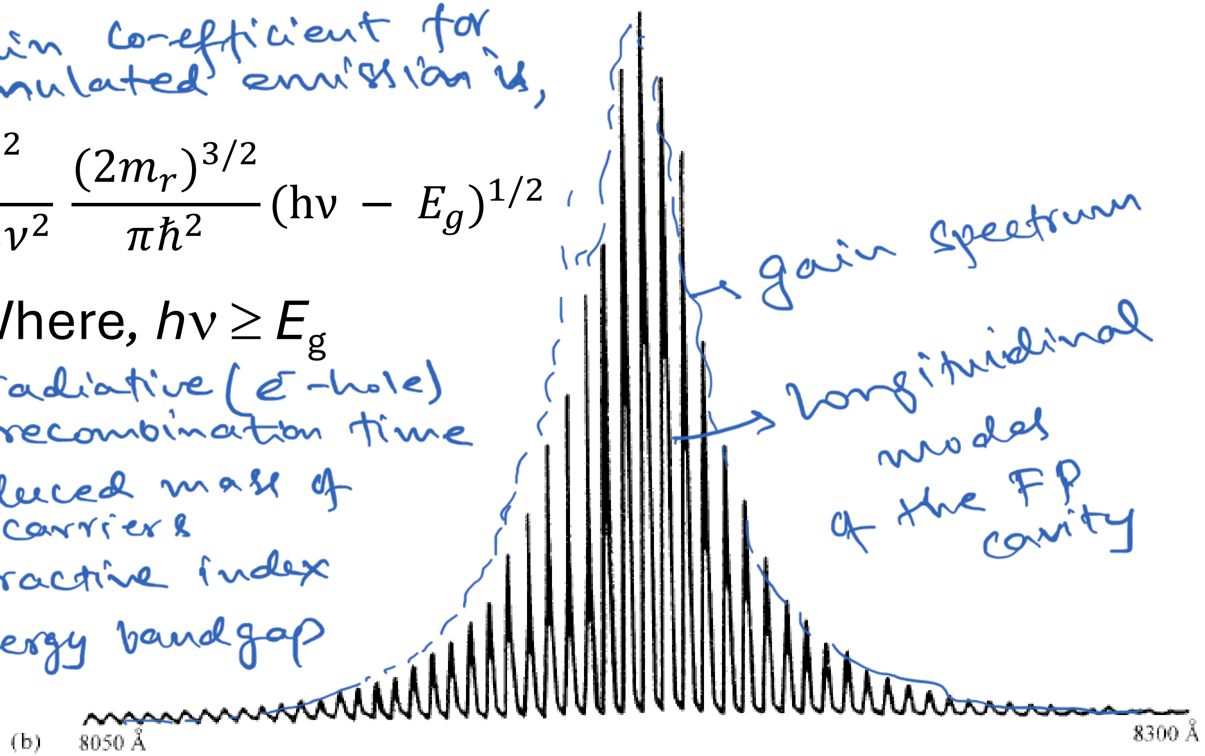
Where, $h\nu \geq E_g$

$\tau_r \rightarrow$ radiative (e^--hole) recombination time

$m_r \rightarrow$ reduced mass of carriers

$n \rightarrow$ refractive index

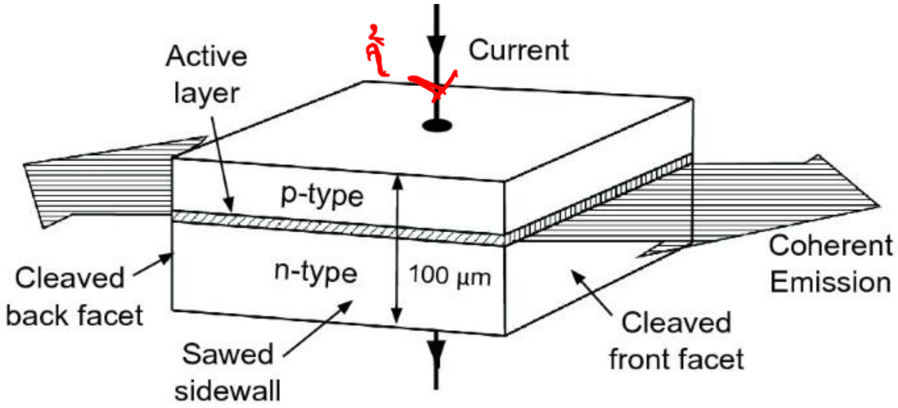
$E_g \rightarrow$ Energy bandgap



→ As the injection current increases the electron-hole recombination gives the spontaneous emission spectrum

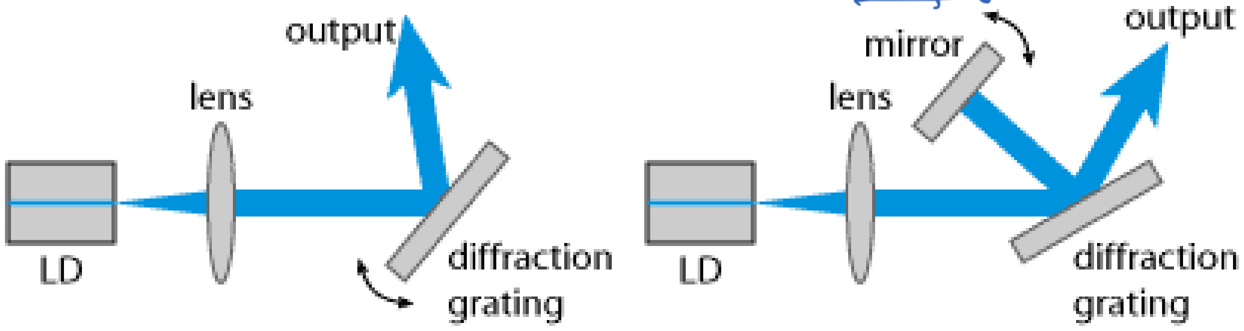
→ At a certain injection current (threshold current) when, $E_{fc} - E_{fv} > E_g$

→ Then the diode starts Lasing!



SINGLE MODE LASING CONFIGURATION

External Cavity Diode Laser (ECDL) configuration



a) Littrow configuration

b) Littman-Metcalf configuration

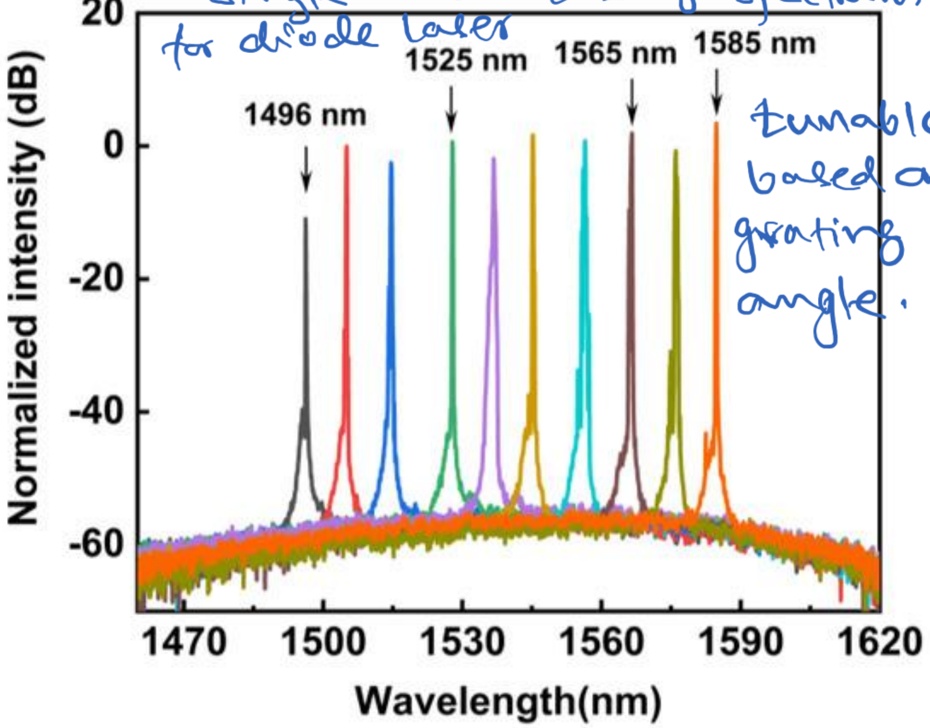
Periodic array of lines (notches) with spacing of the order (multiple) of desired wavelength.

→ It creates diffraction orders for the incident laser beam for a specific wavelength modes that match the diffraction order (condition)

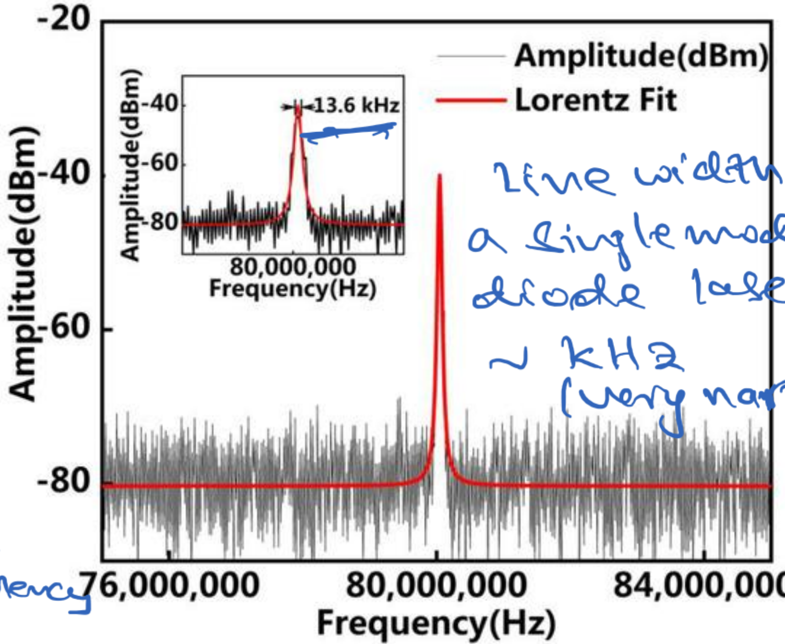
→ Littman-Metcalf configuration

the 1st order diffraction is fed-back to the laser diode to create a cavity feedback → Basically one longitudinal mode is reflected back to the laser diode cavity to achieve frequency or wavelength selection

Single mode lasing spectrum for diode laser

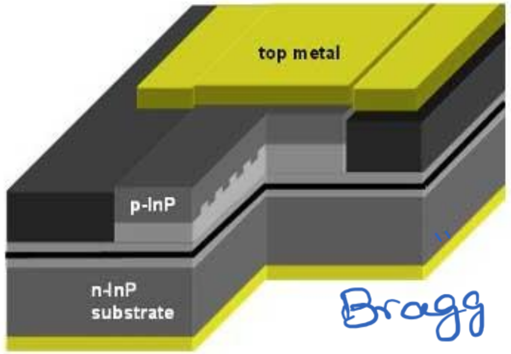


tunable based on grating angle.



Line width of a single mode diode laser ~ kHz (very narrow)

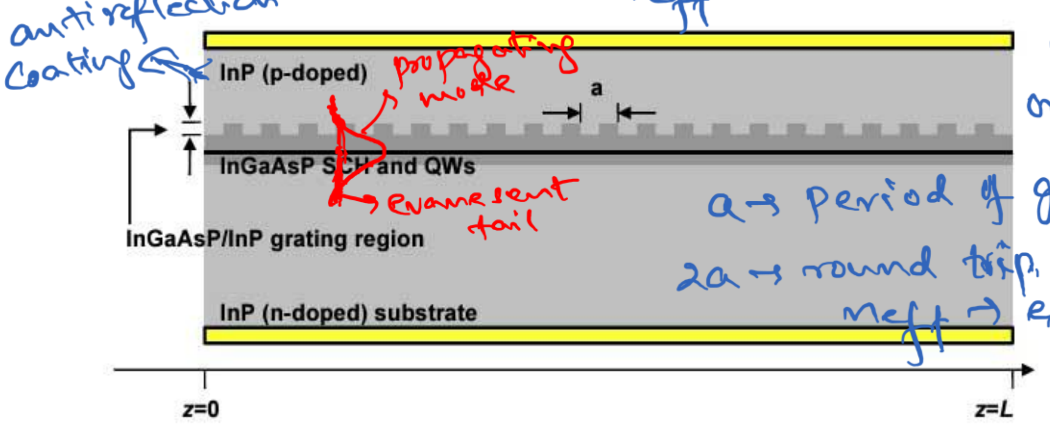
DISTRIBUTED FEEDBACK LASER: SINGLE MODE OPERARTION



DFB LASER

Bragg condition for reflection

$n_{eff} \cdot 2a = m \cdot \lambda_0$

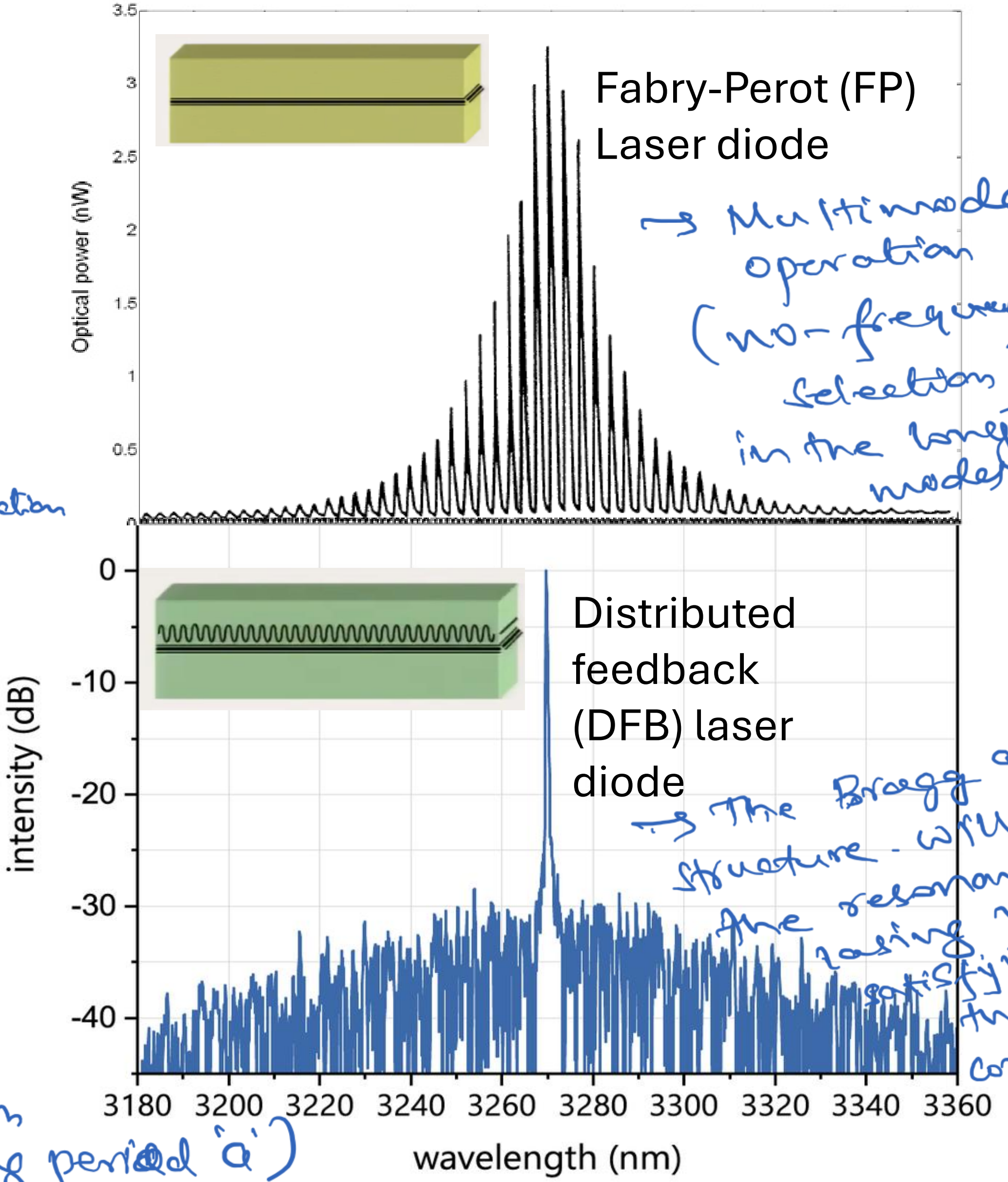


$m = 1, 2, 3, \ldots$ order of diffraction

$a \rightarrow$ period of grating

$2a \rightarrow$ round trip

$n_{eff} \rightarrow$ effective refractive index of bragg grating



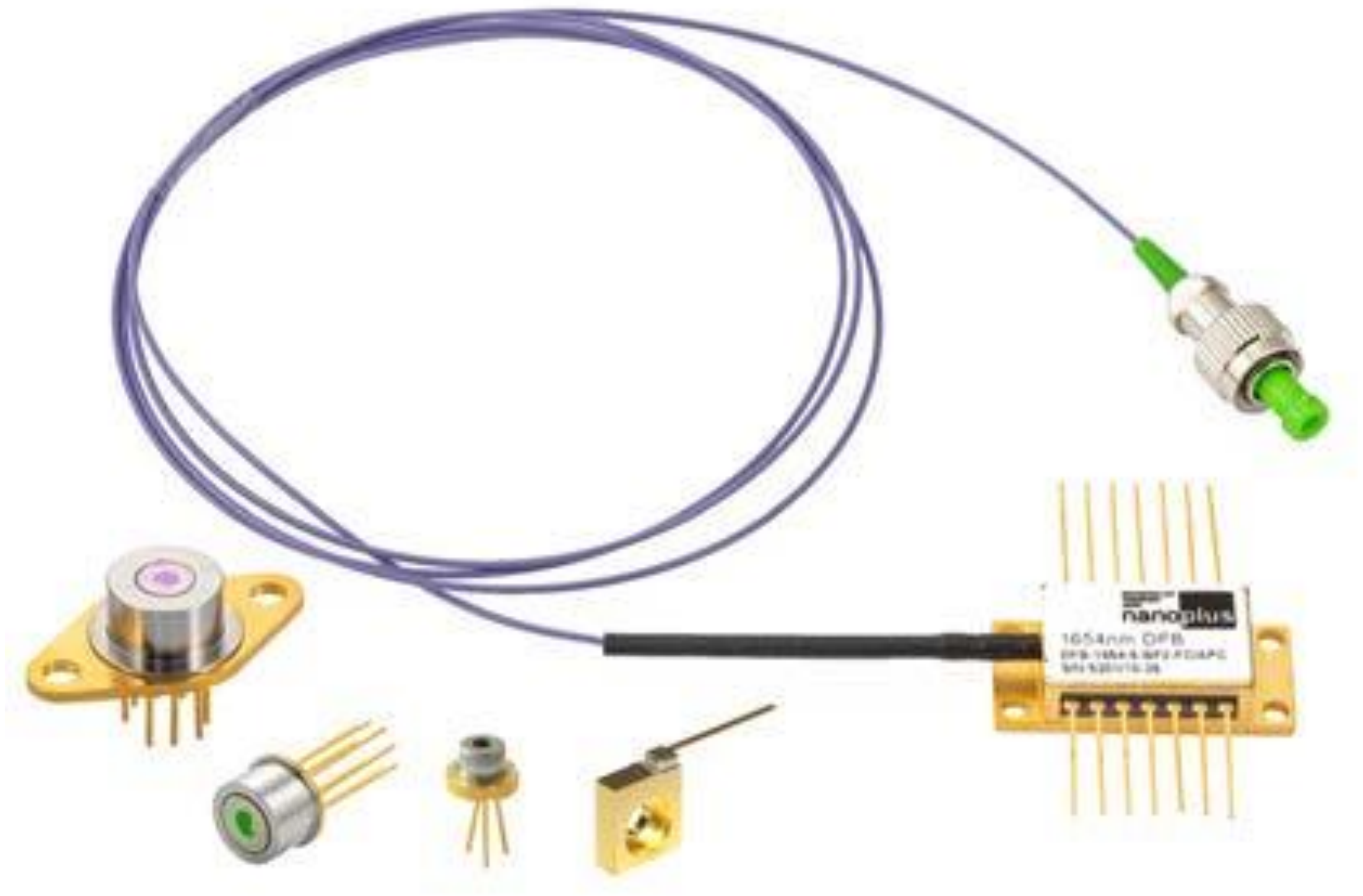
→ Multimode operation (no-frequency selection) in the longitudinal mode

→ The Bragg grating structure will filter the resonant mode the lasing satisfying the Bragg condition

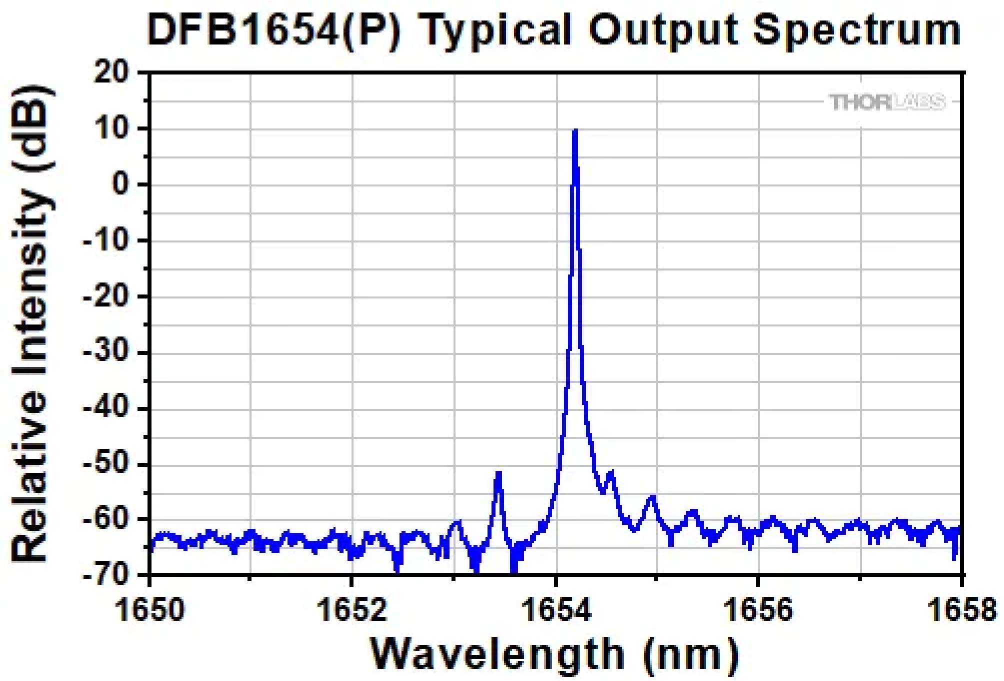
→ Distributed bragg grating along the length of the active medium in the p-n junction.

→ Only one mode satisfying the condition for bragg reflection (matching the spacing period 'a') will be coupled and propagated through active region.

DISTRIBUTED FEEDBACK LASER: SINGLE MODE OPERARTION



DFB laser mode



FIBER LASERS: A Laser with doped Fiber as a gain medium

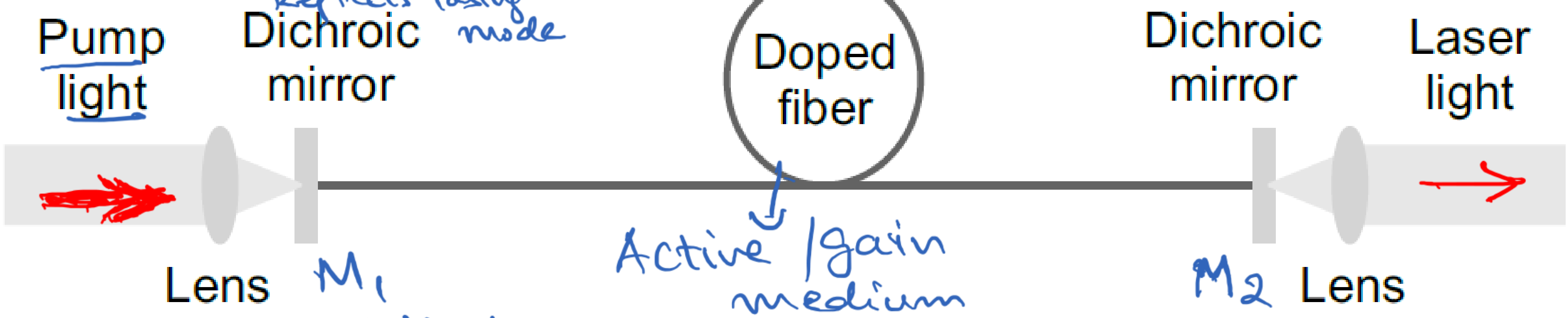
Typically used rare earth elements

Nd → Neodymium
Er → Erbium
Yb → Yttirbium
[Lanthanide series]

Pump
gain medium
Resonator
} → Laser

Transmits pump, Reflects lasing mode

Doped with Rare-earth elements

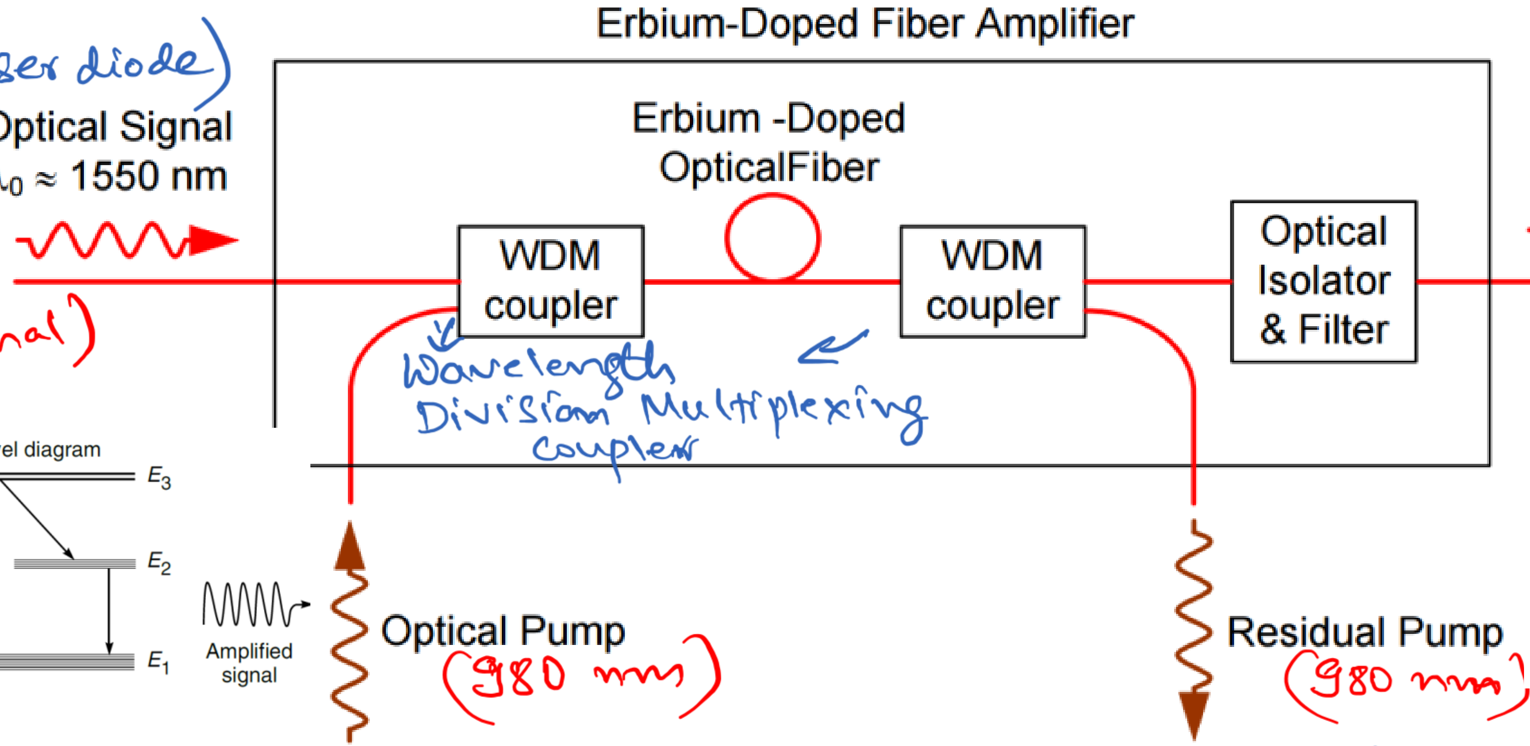


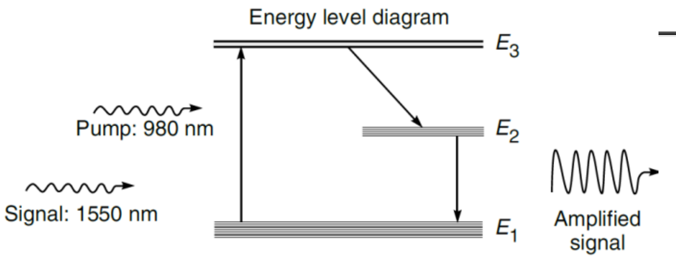
optical feedback

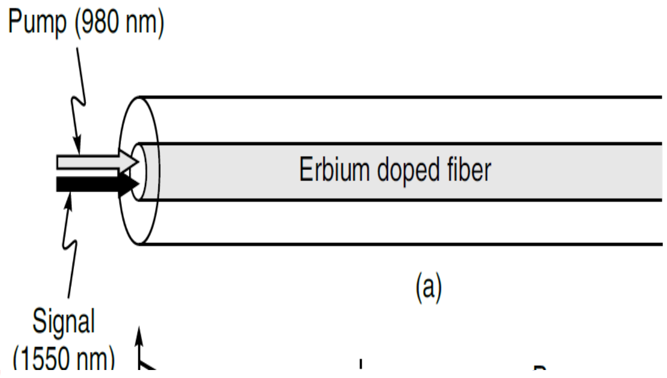
Advantages of Fiber Lasers

- High Efficiency (70-80%) → Due to very low quantum defects in the gain medium
- Very good beam output quality → single mode fibers provide a pure gaussian beam output (no-diffraction)
- CW, and Pulsed Operations (fs lasers available) → ~100 fs pulse width & few MHz rep rate.
- Very high output powers (several kW) → in CW mode operation
- Long Maintenance-free lifetime (~10,000 hrs)
- Tunable Spectrum (Wavelength range: 1um – 2um) → in the Near IR spectral regime (where fiber losses are minimum)
- Optical fiber communication → 1550 nm → Er-doped fiber laser (Erbium)

ERBIUM-DOPED FIBER LASER AMPLIFIER (EDFA):

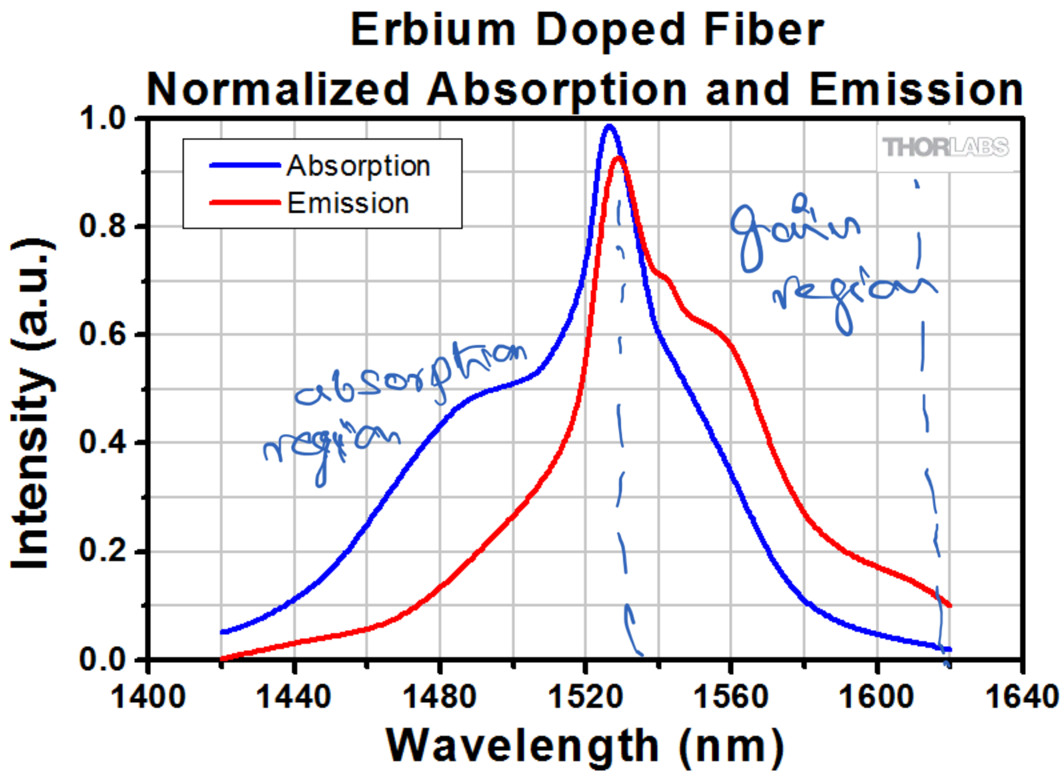


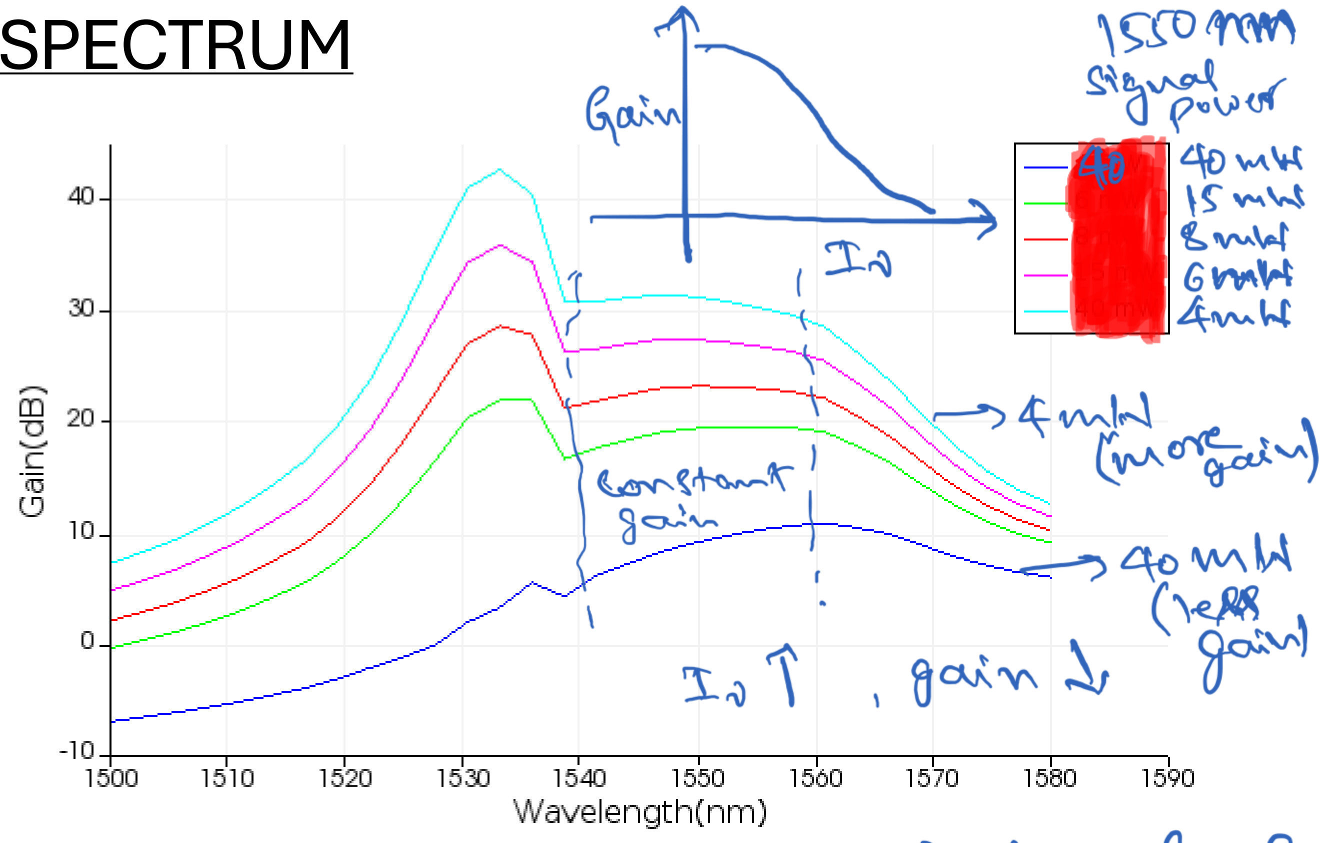




- → Er_2O_3 (Erbium) is doped in $GeO_2 : SiO_2$ core of a fiber (single mode)
- → Concentration ~ 40 – 400 ppm
- → Fiber Numerical aperture → 0.2 (depends on the core and cladding refractive indices)
  ↳ dictates the incidence angle for satisfying TIR inside fiber
- → 3-level Energy system → pulsed lasing.
- → Gain ~ 30 – 40 dB

ERBIUM-DOPED FIBER LASER: GAIN SPECTRUM





Absorption & emission spectrum for Erbium doped fiber

→ Emission spectral width ~ 40nm (broad band emission)

→ Emission > absorption ⇒ gain region

→ Gain spectrum for Erbium doped Laser as the input weak signal power is varied

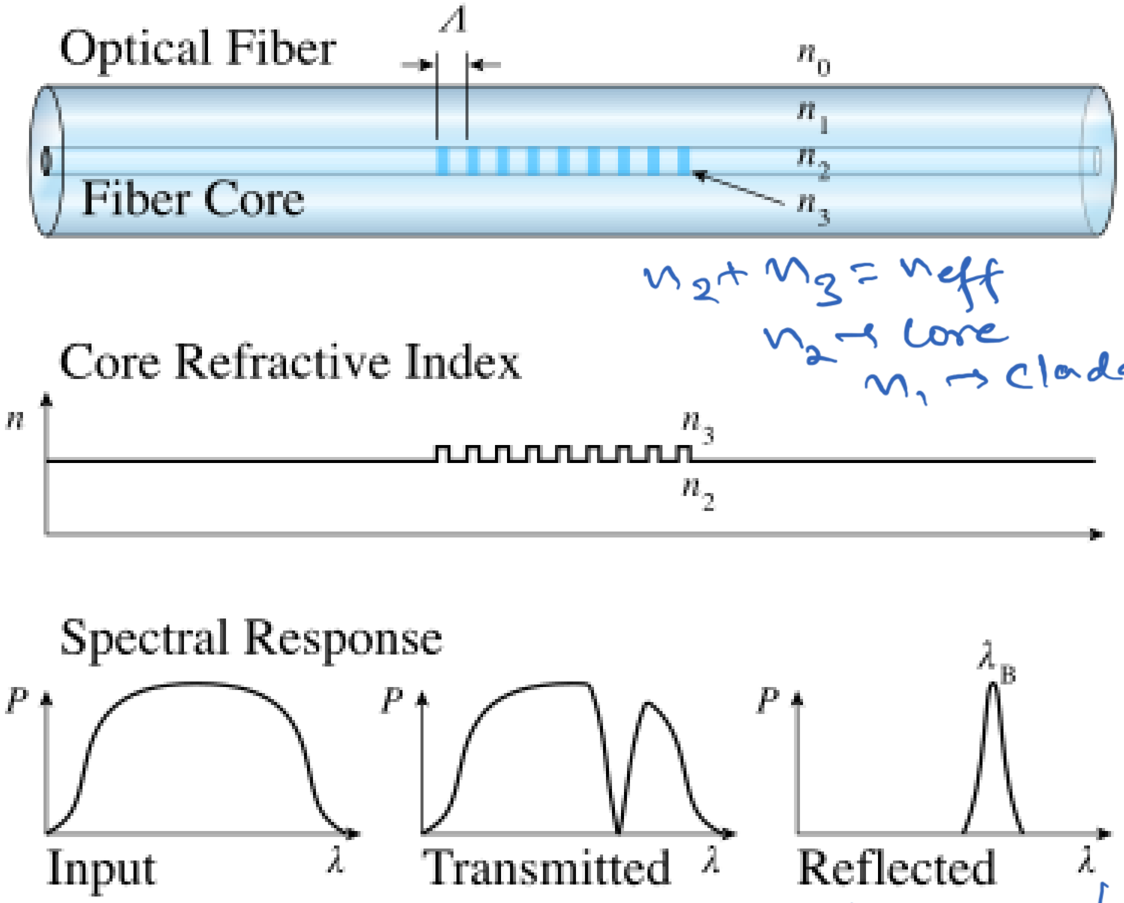
→ For higher input power of the signal beam, the gain amplitude decreases.

saturated gain co-efficient → $\gamma(\nu) = \frac{\gamma_0(\nu)}{1 + I_\nu / I_s}$

I_s → saturation intensity

I_ν → input power

FIBER BRAGG GRATINGS: Fiber Resonators



$n_2 + n_3 = n_{eff}$

$n_2 \rightarrow$ core

$n_1 \rightarrow$ cladding

FBG :- A periodic change in the refractive index creates a grating that reflects the light under the Bragg's condition.

$$\lambda_r = 2 n_{eff} \Lambda$$

$\Lambda \rightarrow$ period (spacing length)

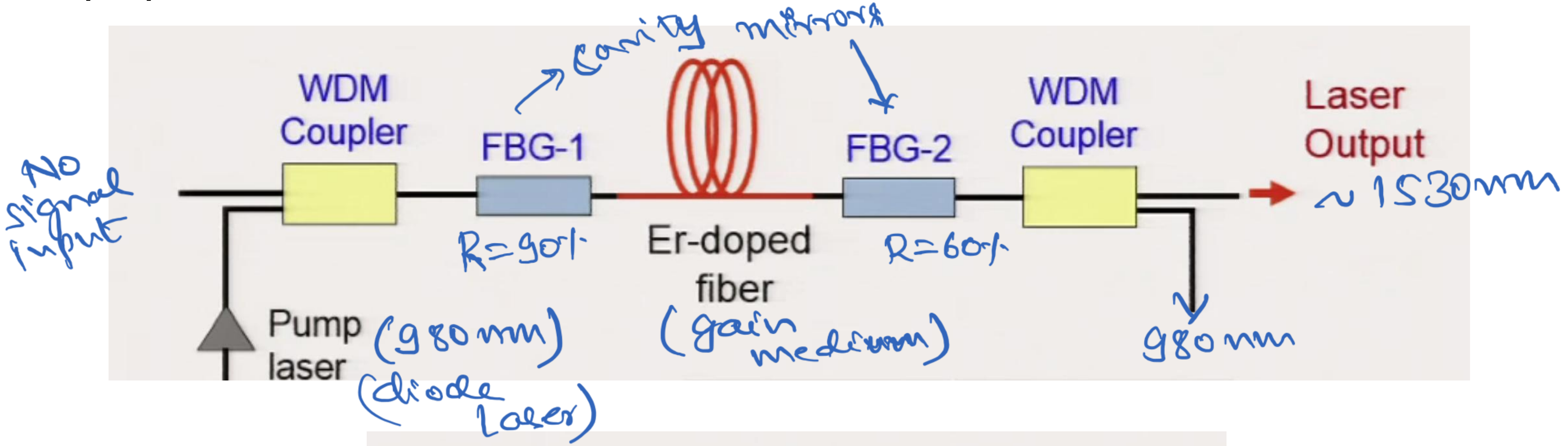
$n_{eff} \rightarrow$ effective refractive index

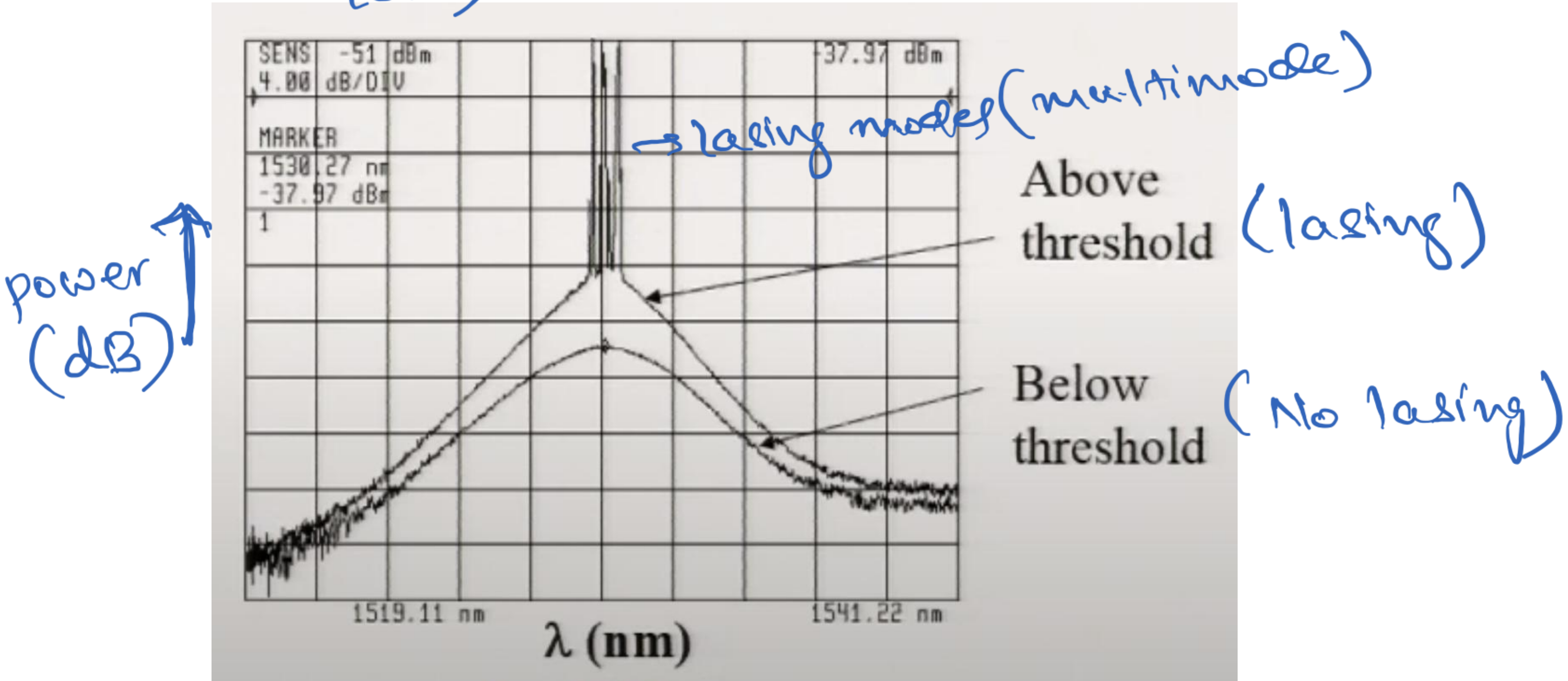
$n_1 < n_{eff} < n_2$

→ Only the wavelength of light that satisfies the Bragg condition resonates with the FBG cavity and provides the lasing action for the single ~~mode~~ mode operation (narrow band)

That couples to the Fiber gain medium to induce lasing action

ERBIUM (Er)-DOPED FIBER LASER

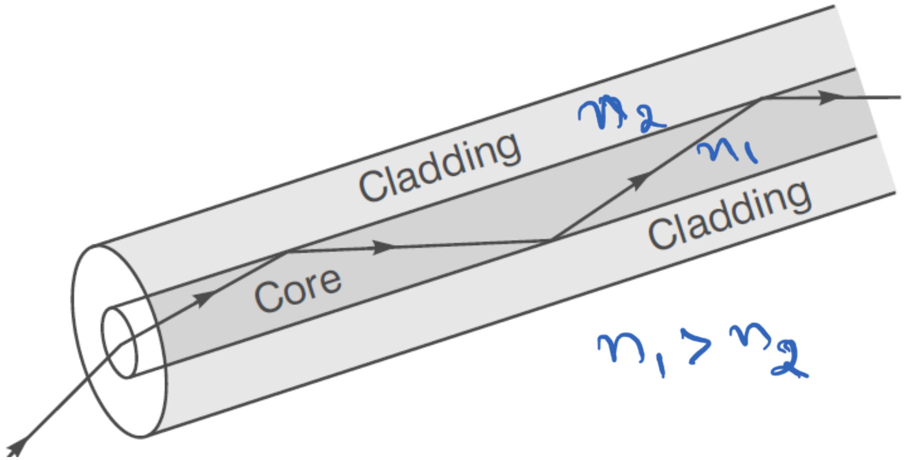


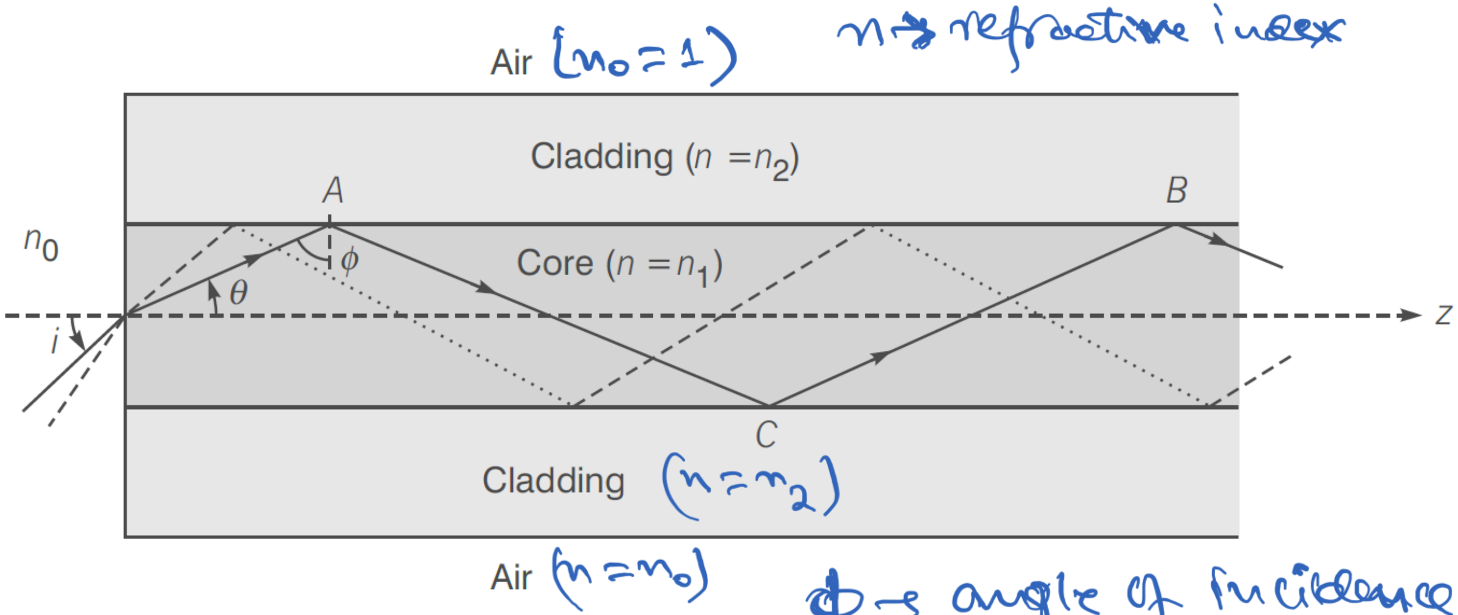


$\lambda_l \sim 1530$ nm

THE OPTICAL FIBER

An optical fiber consists of a central dielectric core, which is cladded by a material of slightly lower refractive in



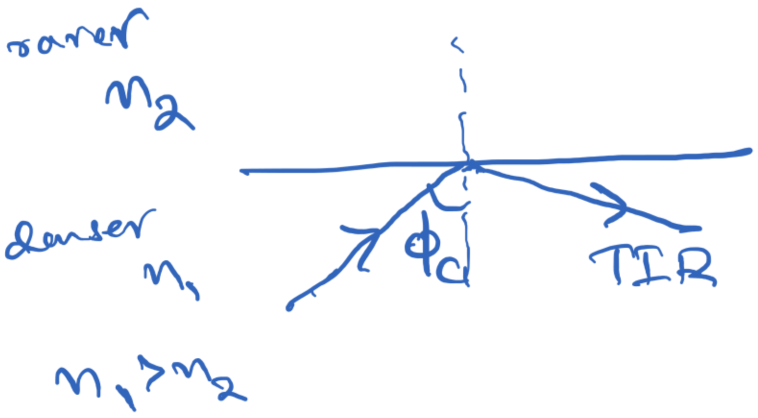


$n \rightarrow$ refractive index

$\phi \rightarrow$ angle of incidence

$\theta \rightarrow$ angle of refraction

Total internal reflection



$$\phi_c = \sin^{-1}\left(\frac{n_2}{n_1}\right)$$

for $\phi > \phi_c \rightarrow$ TIR

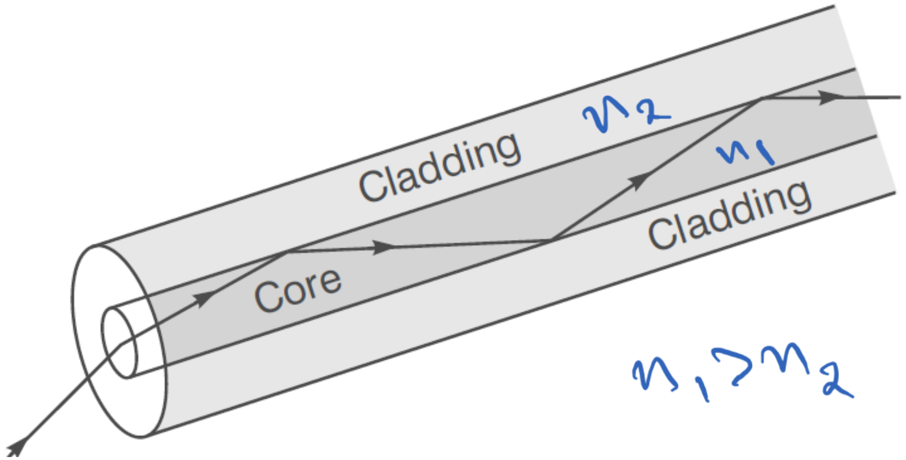
critical angle of incidence for Total Internal Reflection (TIR)

$\rightarrow$ Guides the light through the core-cladding medium (optical fiber)

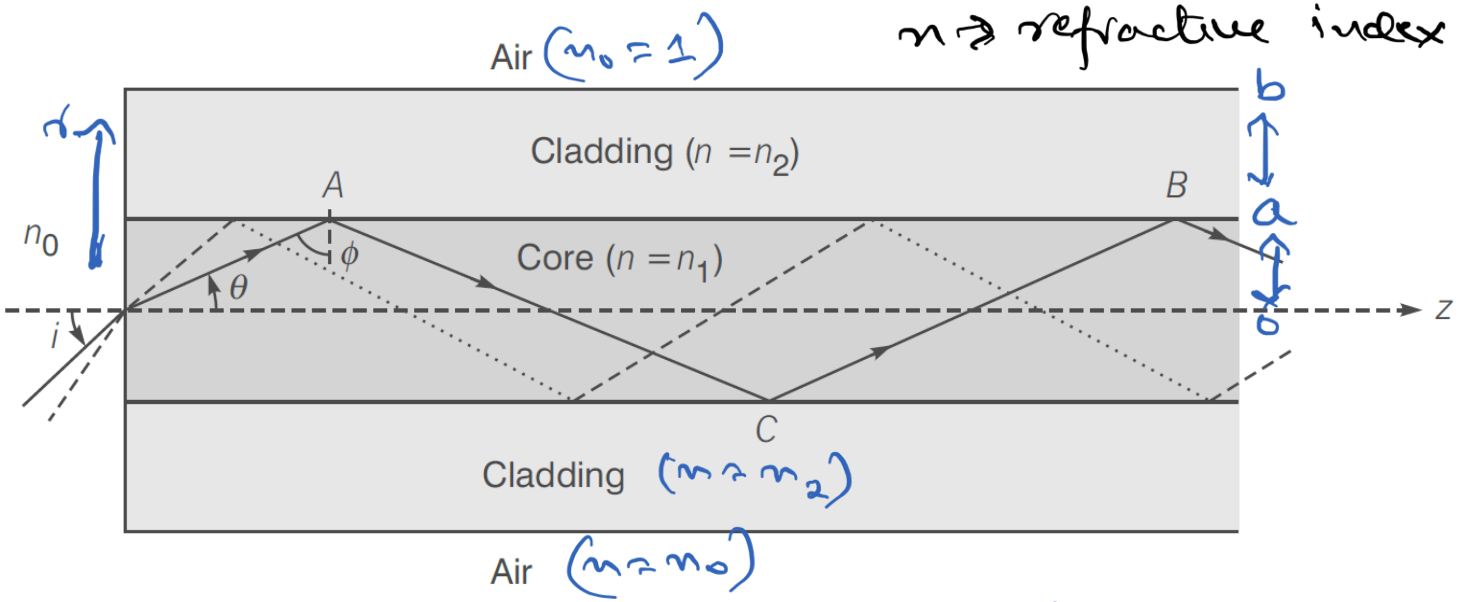
THE OPTICAL FIBER

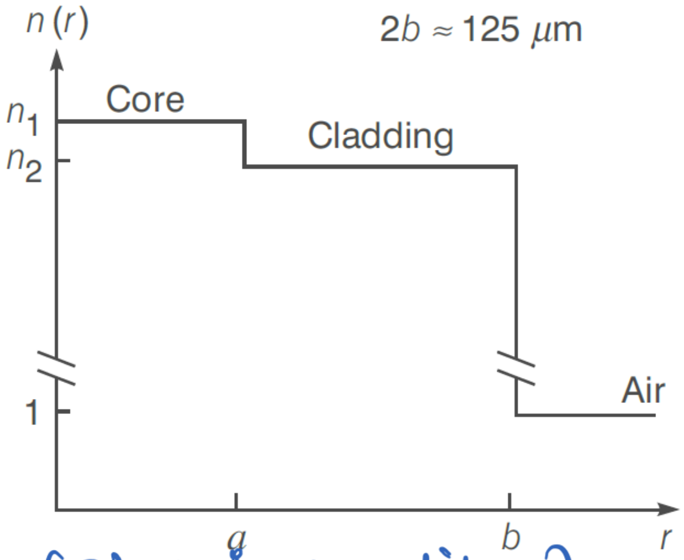
An optical fiber consists of a central dielectric core, which is cladded by a material of slightly lower refractive ir

$n \rightarrow$ refractive index



$n_1 > n_2$





(Step index fiber)

The corresponding refractive index distribution is given by

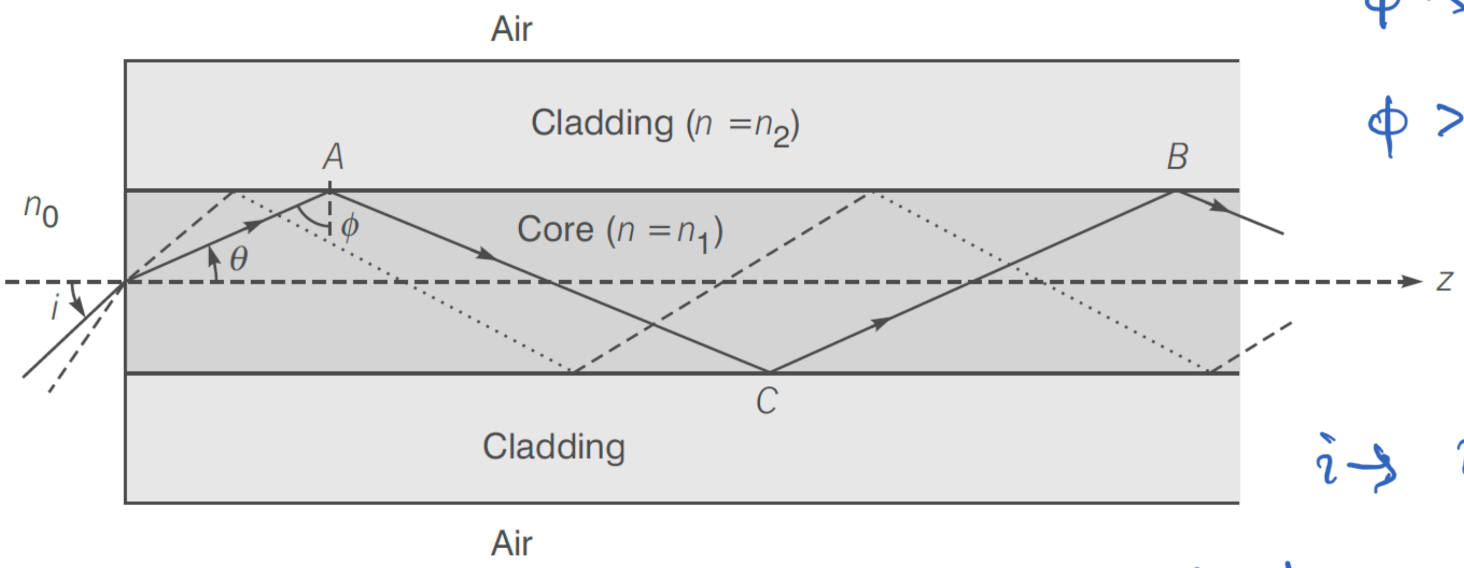
$$n = n_1 \quad \text{for} \quad 0 < r < a$$

$$= n_2 \quad \text{for} \quad a < r < b$$

$n_1 \rightarrow$ refractive index of core

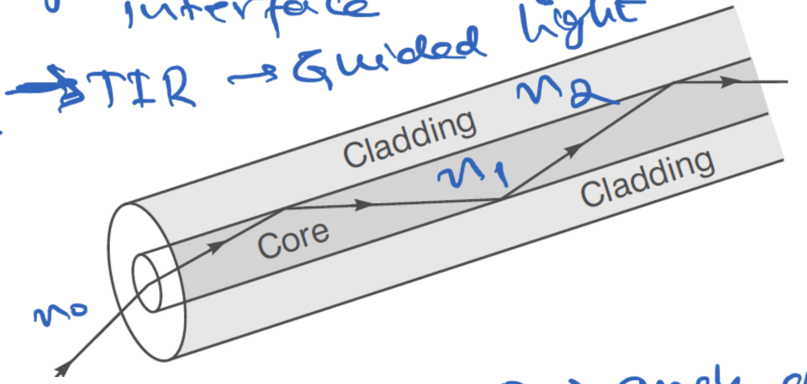
$n_2 \rightarrow$ refractive index of cladding

THE NUMERICAL APERATURE



$\phi \rightarrow$ angle of incidence at n_1 & n_2 interface

$\phi > \phi_c \rightarrow$ TIR $\rightarrow$ Guided light



$i \rightarrow$ Launching angle, $\theta \rightarrow$ angle of refraction.

Lets consider the light is incident on the fiber input interface at an angle 'i' and 'θ' is the angle of refraction, then by Snell's law

$$\frac{\sin i}{\sin\theta} = \frac{n_1}{n_0}$$

$$\theta = \pi/2 - \phi$$

$$\Rightarrow \quad n_0 \sin i = n_1 \sin\theta$$

$$= n_1 \sin(\pi/2 - \phi)$$

$$= n_1 \cos\phi$$

THE NUMERICAL APERATURE

The critical angle $\phi_c = \sin^{-1}(n_2/n_1)$

$\Rightarrow \sin\phi_c = n_2/n_1$

or $\cos^2\phi_c = 1 - n_2^2/n_1^2$

or $\cos\phi_c = \sqrt{1 - n_2^2/n_1^2}$

$n_0 \sin i_m = n_1 \cos\phi_c$ at $\phi = \phi_c$ (TIR)

$\phi > \phi_c$

$$= n_1 \left(\sqrt{\frac{n_1^2 - n_2^2}{n_1^2}} \right)$$

$$\boxed{n_0 \sin i_m = \sqrt{n_1^2 - n_2^2}}$$

$n_0 = 1$ (air)

$$\boxed{\sin i_m = \sqrt{n_1^2 - n_2^2}} \rightarrow \text{Numerical aperture (NA)}$$

$i_m \rightarrow$ maximum acceptance angle

$$i_m = \sin^{-1}\left(\sqrt{n_1^2 - n_2^2}\right)$$

→ defines the maximum angle of light incidence to ensure the efficient light coupling to the fiber to achieve TIR.

THE NUMERICAL APERATURE

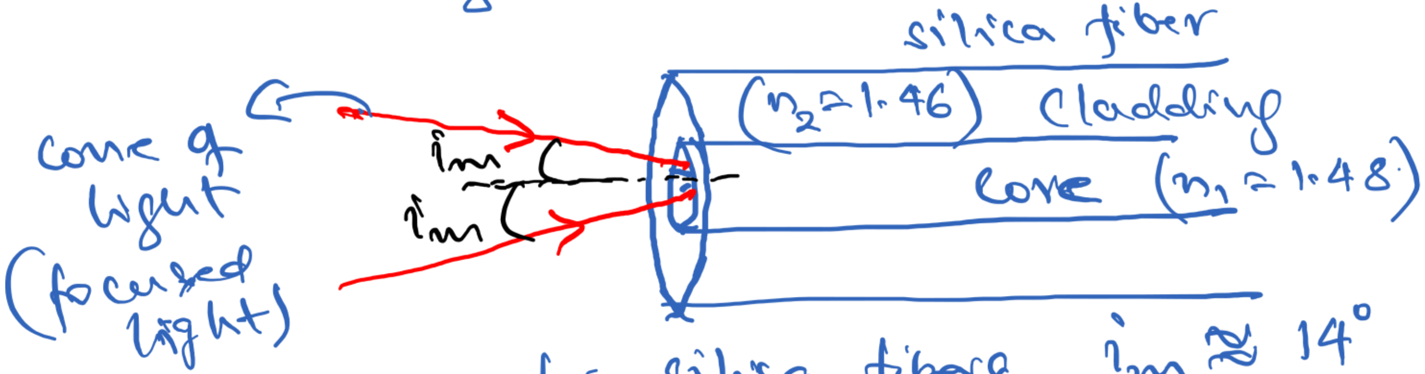
So, in general $\sin i < \left(\frac{n_1^2 - n_2^2}{n_0^2}\right)^{1/2}$ for efficient light coupling and guiding in the fiber

if $(n_1^2 - n_2^2) \geq n_0^2$, then for all '$i$', total internal reflection will occure at the core-cladding interface. (larger NA → leads to larger launching angle)

or $\sqrt{n_1^2 - n_2^2} \geq n_0 \implies$

$NA \geq n_0$

→ If a cone of light is incident on one end of the fiber, it will be guided through the fiber provided the semi angle of the cone is less than i_m

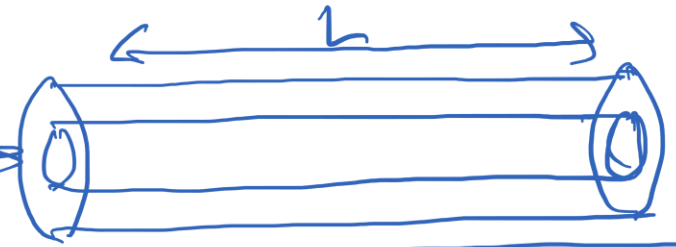


for silica fibers $i_m \approx 14°$

$i_m \leq \sin^{-1}\left(\sqrt{n_1^2 - n_2^2}\right)$

→ TIR and light guiding through the fiber

ATTENUATION IN OPTICAL FIBERS



(laser LED) P_{in} → input power

P_{out} (photodetector) measured using → output power

$$\boxed{\text{loss (dB)} = 10 \log\left(\frac{P_{in}}{P_{out}}\right)}$$

$$\boxed{P_{out} < P_{in}}$$ (attenuation losses)

→ losses are quantified in decibles (dB)

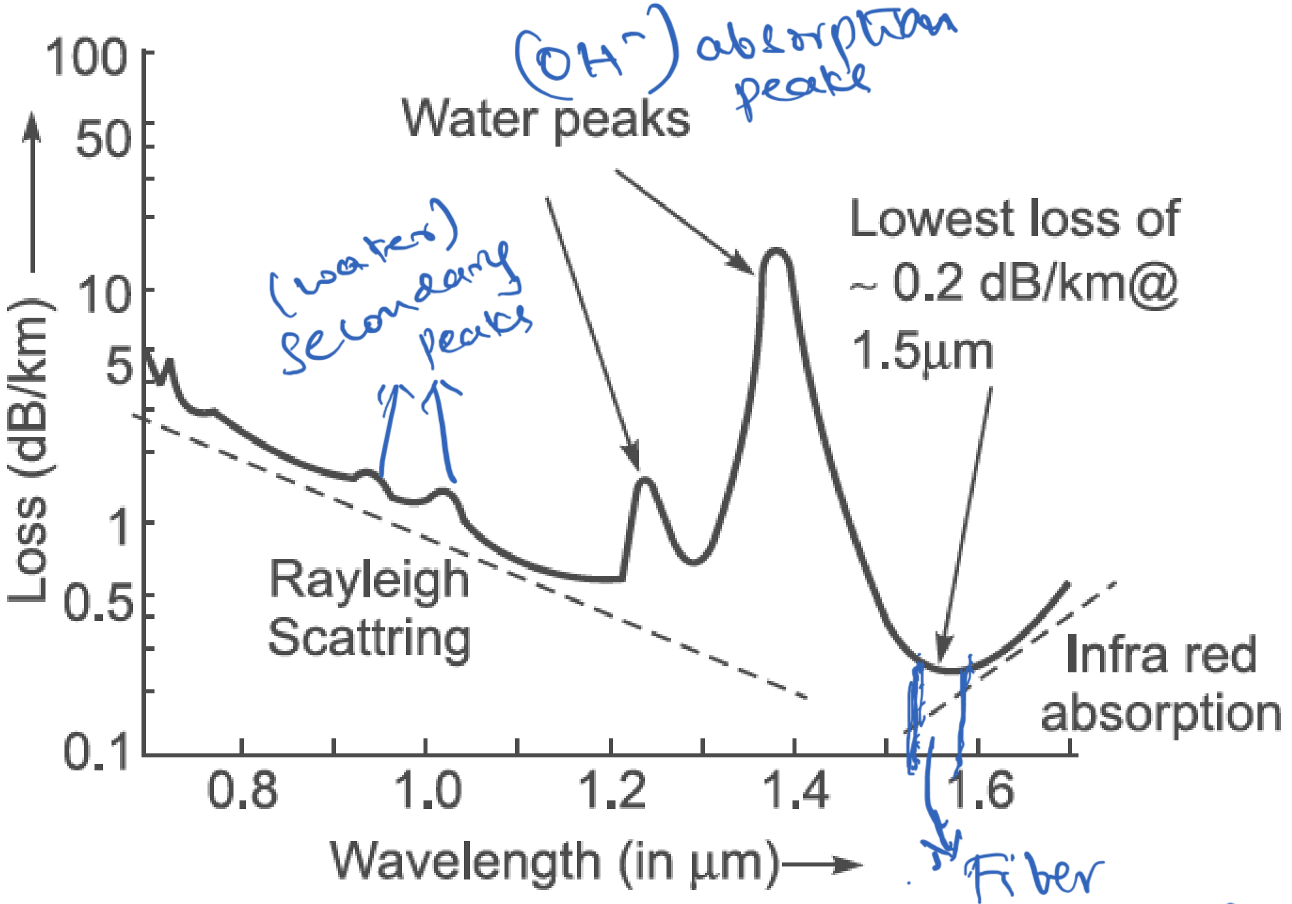
0 dB → $P_{in} = P_{out}$ → no losses

Loss Mechanisms

- Absorptive
  - → Electronic / vibrational transitions (intrinsic losses)
  - → Impurities (OH⁻ content) water vapor (Extrensic losses due to fabrication)
- Radiative
  - dopants
  - → Rayleigh scattering losses → (Guided light gets couples to the radiation mode)
  - → Reduces the TIR

Scattering $\propto 1/\lambda_{light}^4$

ATTENUATION IN OPTICAL FIBERS



Fiber communication wavelength (1550nm) (C- band) → very low attenuation losses (0.2 dB/km)

(Silica based optical fiber)

Other losses

(1) Bending losses

$$\text{loss} \propto e^{-(R/R_c)}$$

R → Radius of curvature, where, $R_c = \dfrac{a}{\sqrt{n_1^2 - n_2^2}}$

~ 0.2–4 μm

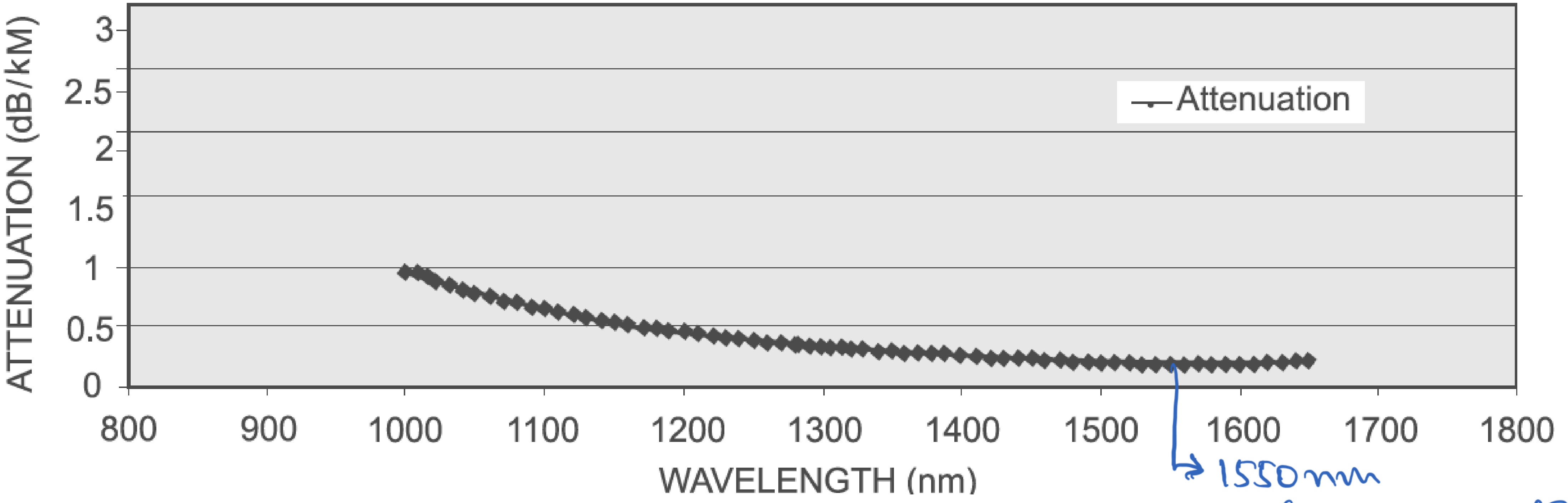
(2) Connector losses

(3) insertion losses

(4) Thermal effects

ATTENUATION IN OPTICAL FIBERS

Ref: Ajoy Ghatak, Optics (Sterlite Industries, India)



→ 1550 nm
(loss ~ 0.2 dB/km)

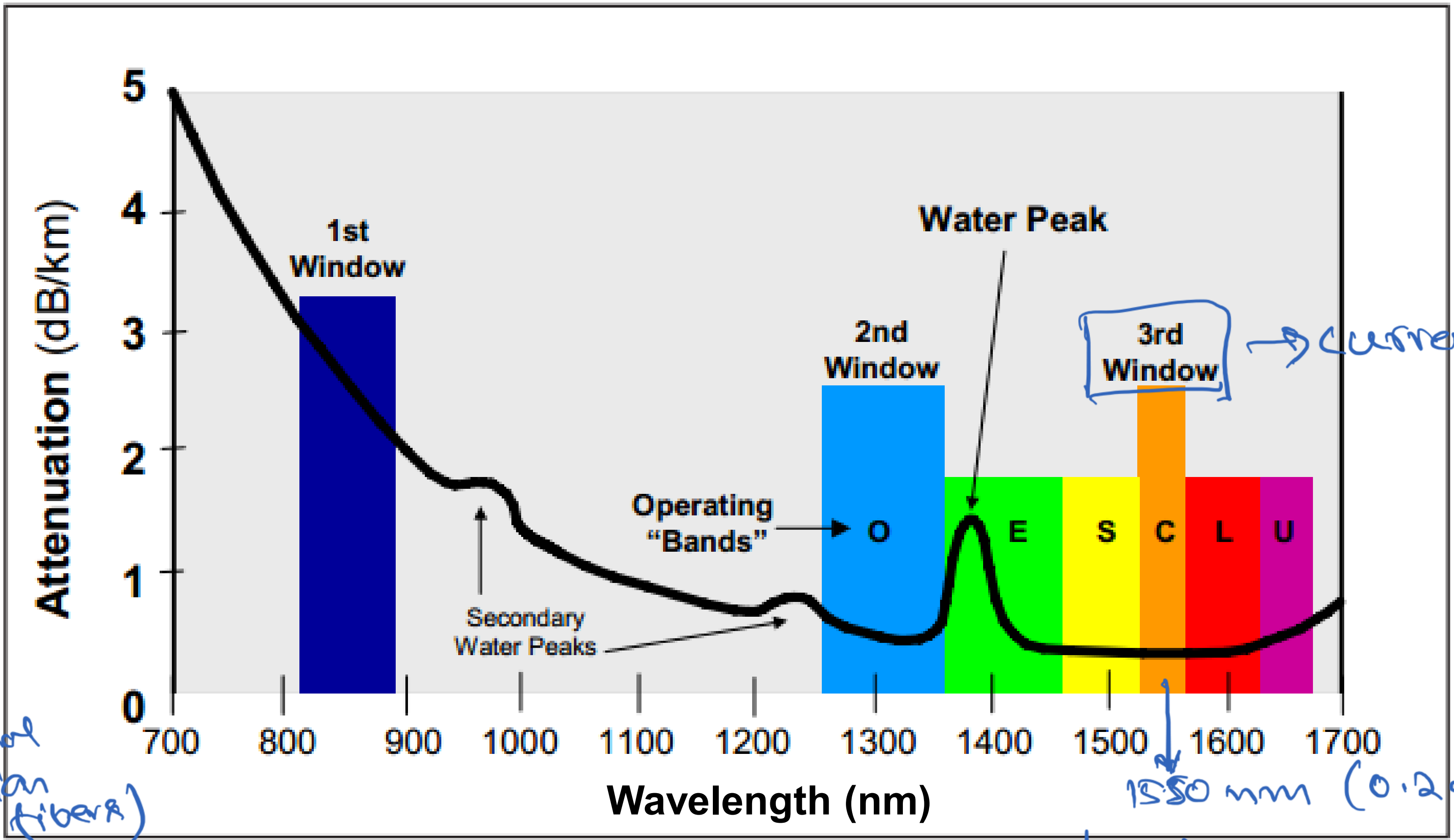
→ $\text{Loss (dB)} = -10 \log\left(\frac{P_{out}}{P_{in}}\right)$, if $P_{out} = \frac{1}{2} P_{in}$ (50% loss)

$\Rightarrow -10 \log\left(\frac{1}{2}\right) \Rightarrow 10 \log 2 = 10 \times 0.3 = 3\,\text{dB}$ (loss)

The distance propagated for 3dB loss $= \frac{3\,\text{dB}}{0.2\,\text{dB/km}} = 15\,\text{km}$

⇒ For every 15 km, the propagation power (signal) reduces by half.

OPERATING TELECOMMUNICATION BANDS



→ current fiber optic communication band. (1530 nm – 1560 nm)

1550 nm (0.2dB/km)

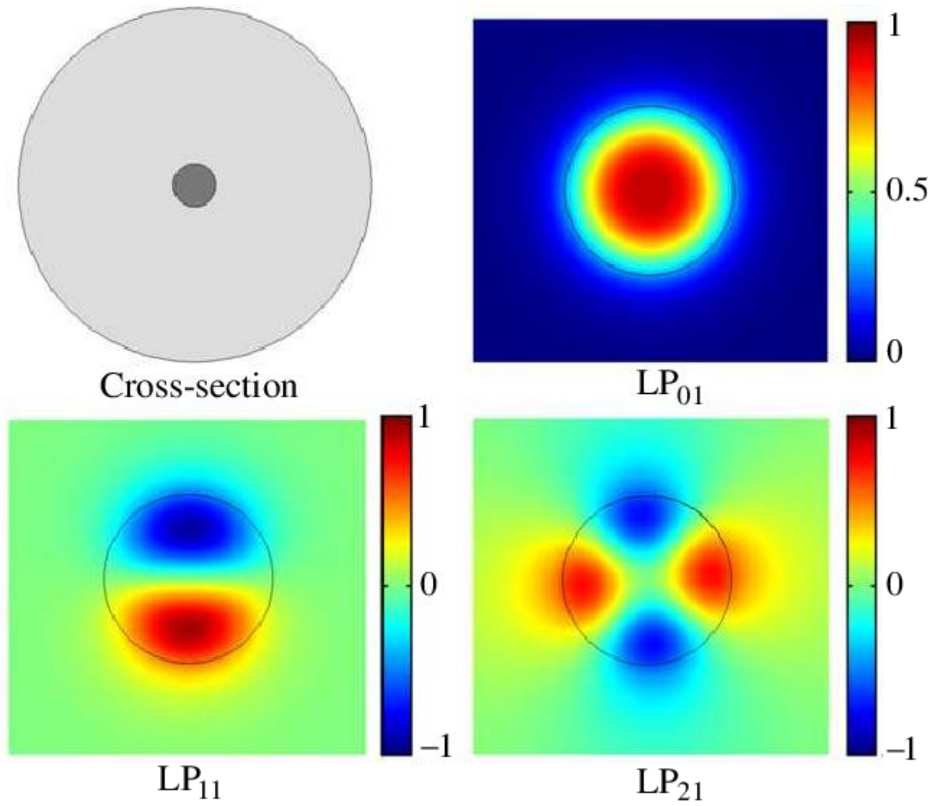
(Favors long distance signal transmission through fibers)

→ Due to low attenuation loss (0.2dB/km) @ 1550 nm, this wave length is currently preffered for fiber-optical communication

→ Availability of Er-doped fiber amplifiers (1530 nm – 1560 nm)

‘MODES’ IN OPTICAL FIBERS

- The light propagation in Fibers is determined by the Maxwells equations
- The boundary conditions at the interface dictate the allowed modes in a Fiber
- Transverse Modes: Allowed field distribution



LP → transverse profiles of the modes. in a fiber →

Normalized Waveguide Parameter

$$V = \frac{2\pi}{\lambda_0} a \sqrt{n_1^2 - n_2^2}$$

a : core radius
n_1: core refractive index
n_2: cladding refractive index
λ_0 : wavelength of light

$0 < V < 2.41$: Single mode (LP_{01}: fundamental mode)

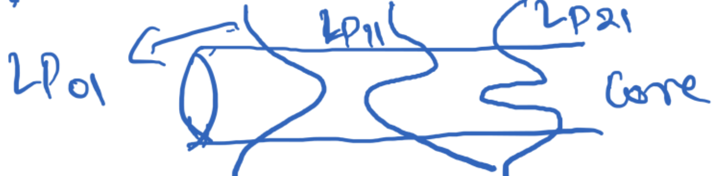
$2.41 < V < 3.83$: Multimode (LP_{01}, LP_{11})

$3.83 < V < 5.14$: Multimode (LP_{01}, LP_{02}, LP_{11}, LP_{21})

…

LP → linear polarized modes

01 → indices (l, m) → orders of Bessel's solns.



'MODES' IN OPTICAL FIBERS

The bound values for 'V' dictates the cut-off wavelength for a fiber to operate in a single mode,

$$\lambda_{cutoff} = \frac{2\pi}{2.41} a \sqrt{n_1^2 - n_2^2} \quad ; \quad \lambda > \lambda_{cut\text{-}off} \rightarrow \text{single mode fiber}$$

<u>Example</u>:- consider a step index fiber with $n_1 = 1.48$ and $n_2 = 1.46$ and core radius $a = 2\ \mu m$

then $$\lambda_{cutoff} = \frac{2\pi}{2.41} \times 2 \times \sqrt{(1.48)^2 - (1.46)^2}\ \mu m$$

$$= \frac{2\pi}{2.41} \times 2 \times 0.24\ \mu m$$

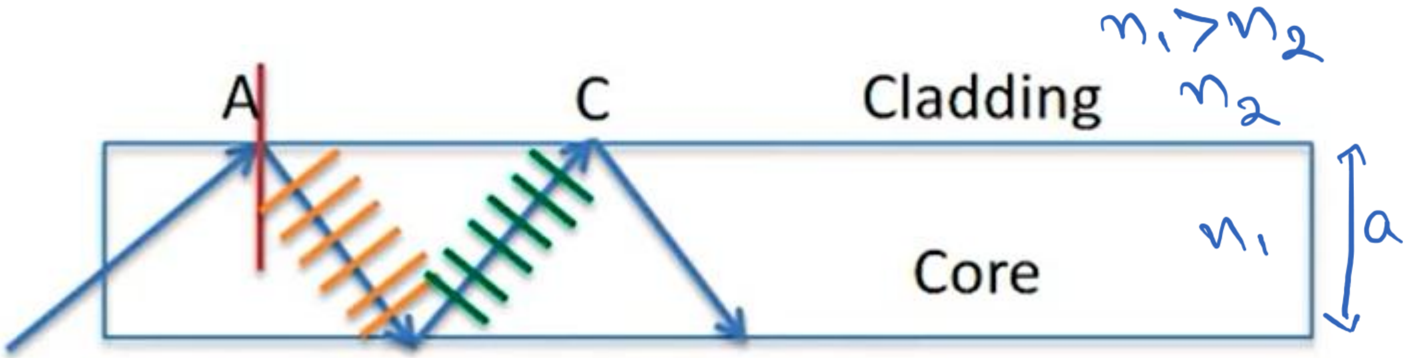
$$\lambda_{cutoff} \simeq 1.26\ \mu m$$

$\Rightarrow$ $\lambda_0 > 1.26\ \mu m$, we will have the L_{01} (single mode)

MULTIMODE AND SINGLE MODE FIBERS

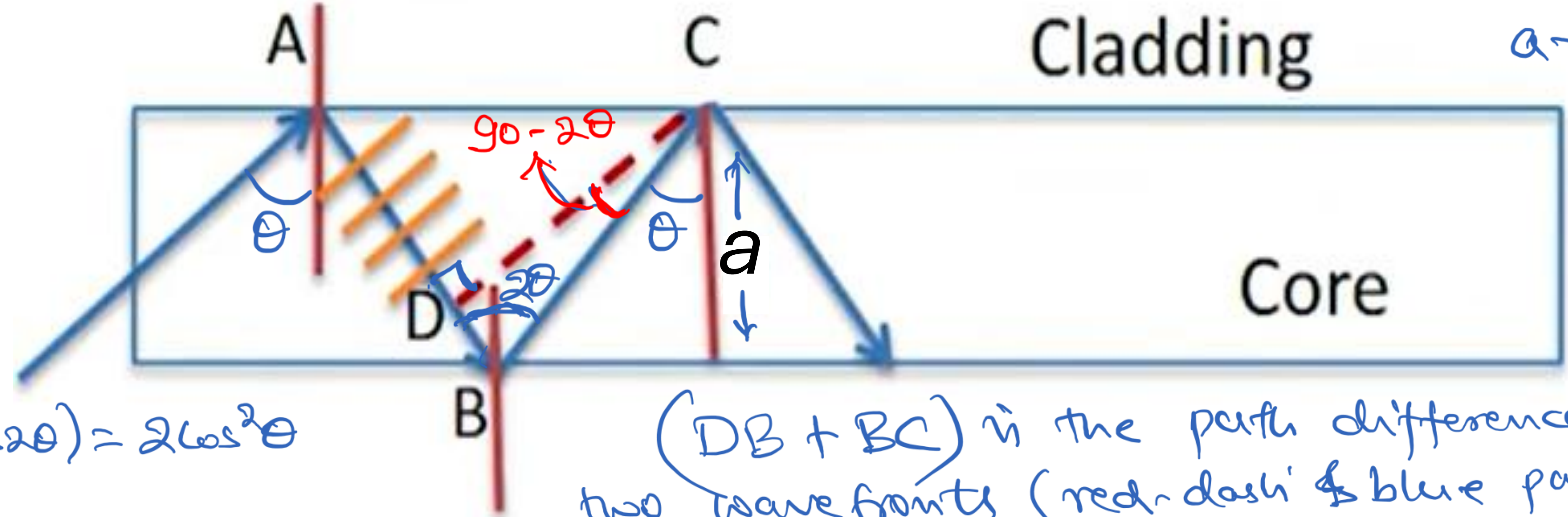
→ step-index fibers

$$n = \begin{cases} n_1 & 0 < a < r \\ n_2 & a < r < b \end{cases}$$



→ single mode propagation of light with constant wavefronts

a → core diameter



$(1+\cos 2\theta) = 2\cos^2\theta$

$(DB + BC)$ is the path difference b/n the two wavefronts (red-dash & blue path)

$BD = BC \sin(90-2\theta)$; $BC = \frac{a}{\cos\theta} = a\sec\theta$

$= BC\cos(2\theta)$

$= a\sec\theta\cos(2\theta)$

$\Rightarrow (BD + BC) = a\sec\theta(1+\cos 2\theta)$

$= a\sec\theta(2\cos^2\theta) = 2a\cos\theta$

MULTIMODE AND SINGLE MODE FIBERS

→ Condition for the wave guiding

⇒ constructive interference of two paths at point 'C'

⇒ phase difference = $2m\pi$

$m = 0, 1, 2, \ldots$

$$\Rightarrow \frac{2\pi}{\lambda}(BD + BC) = 2m\pi$$

or

$$\frac{2\pi}{\lambda}(2a\cos\theta) = 2m\pi$$

or

$$\boxed{2a\cos\theta = m\lambda}$$

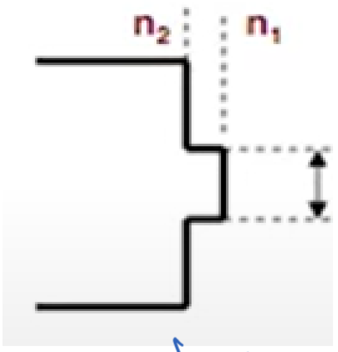
For a given 'a' and 'λ', there are allowed descrete angles θ_m corresponding to the integer multiple of λ.

⇒ there can be multiple transverse modes allowed in a fiber depending on the angle θ_m.

$(\lambda/2, \lambda, 3\lambda/2, \ldots)$.

MULTIMODE AND SINGLE MODE FIBERS

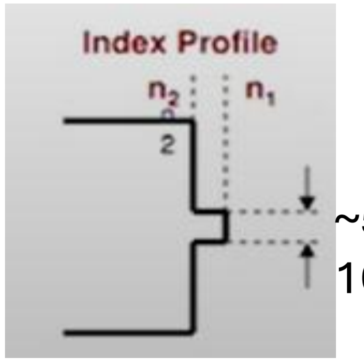
→ (Step index fibers)



(Step-index fiber)

normalised waveguide parameter $V > 2.41$

Supports multiple laser modes in a single propagation (band of wavelengths)



$V < 2.41$

Supports the guiding of a 'one' mode (fundamental mode) throughout the propagation

DISPERSION COMPENSATION IN FIBERS

The Dispersion co-efficient $D = \frac{\Delta\tau}{L\,\Delta\lambda_0}$

Dispersion

$$\Delta\tau = \frac{d\tau}{d\lambda_0}\,\Delta\lambda_0 \quad \text{ps/km.nm}$$

or $(\Delta\tau = D \cdot L \cdot \Delta\lambda_0)$

$L \rightarrow$ Length of propagation

$\Delta\tau \rightarrow$ time delay (pulse dispersion)

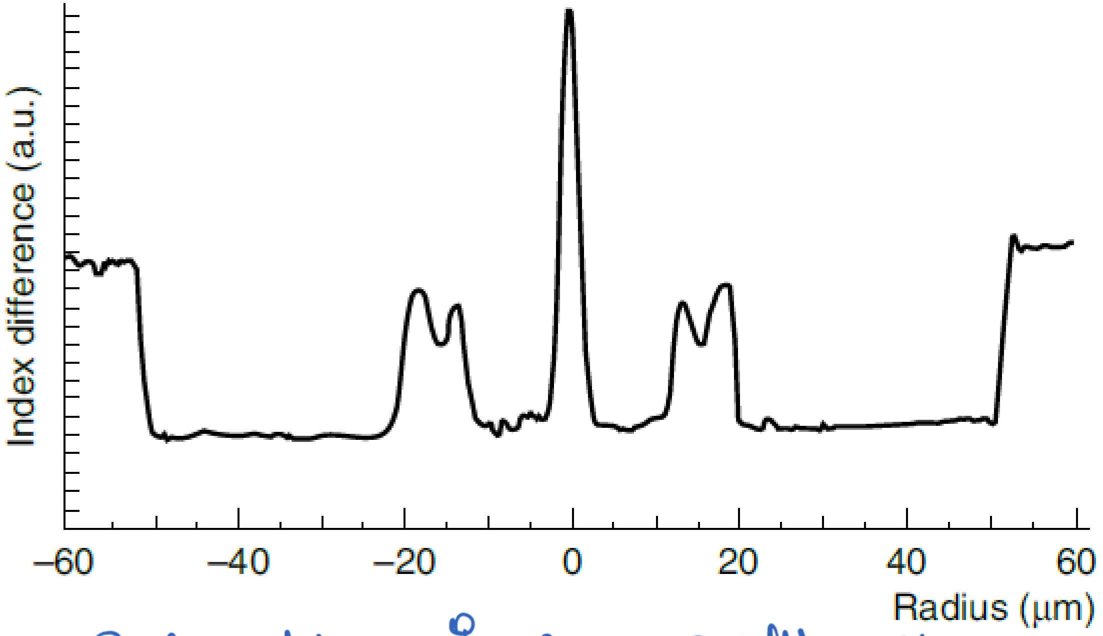
$\Delta\lambda_0 \rightarrow$ spectral width in free space

@ 1.27 µm → zero dispersion but larger attenuation losses (Not ideal/favorable for long distance communication

@ 1.55 µm → low propagation losses but negative dispersion $\left(\frac{dn}{d\lambda} < 0\right)$ (pulse broadening)

$\Delta\tau \simeq 2$ ps/km.nm

⇒ Employs dispersion compensation mechanism to negate the intrinsic dispersion.

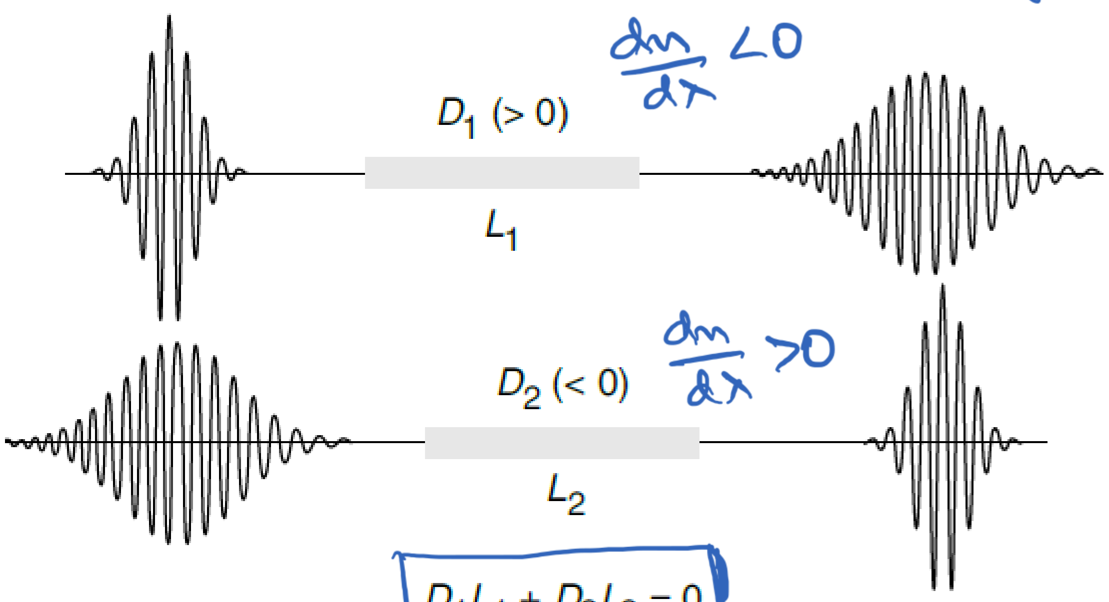


Refractive index profile of a Dispersion compensation fiber (n_f is changed in the core) to compensate dispersion in the signal pulses)

DISPERSION COMPENSATION IN FIBERS

Attaching the fibers of negative and positive dispersion fibers together.

Principle of dispersion compensation

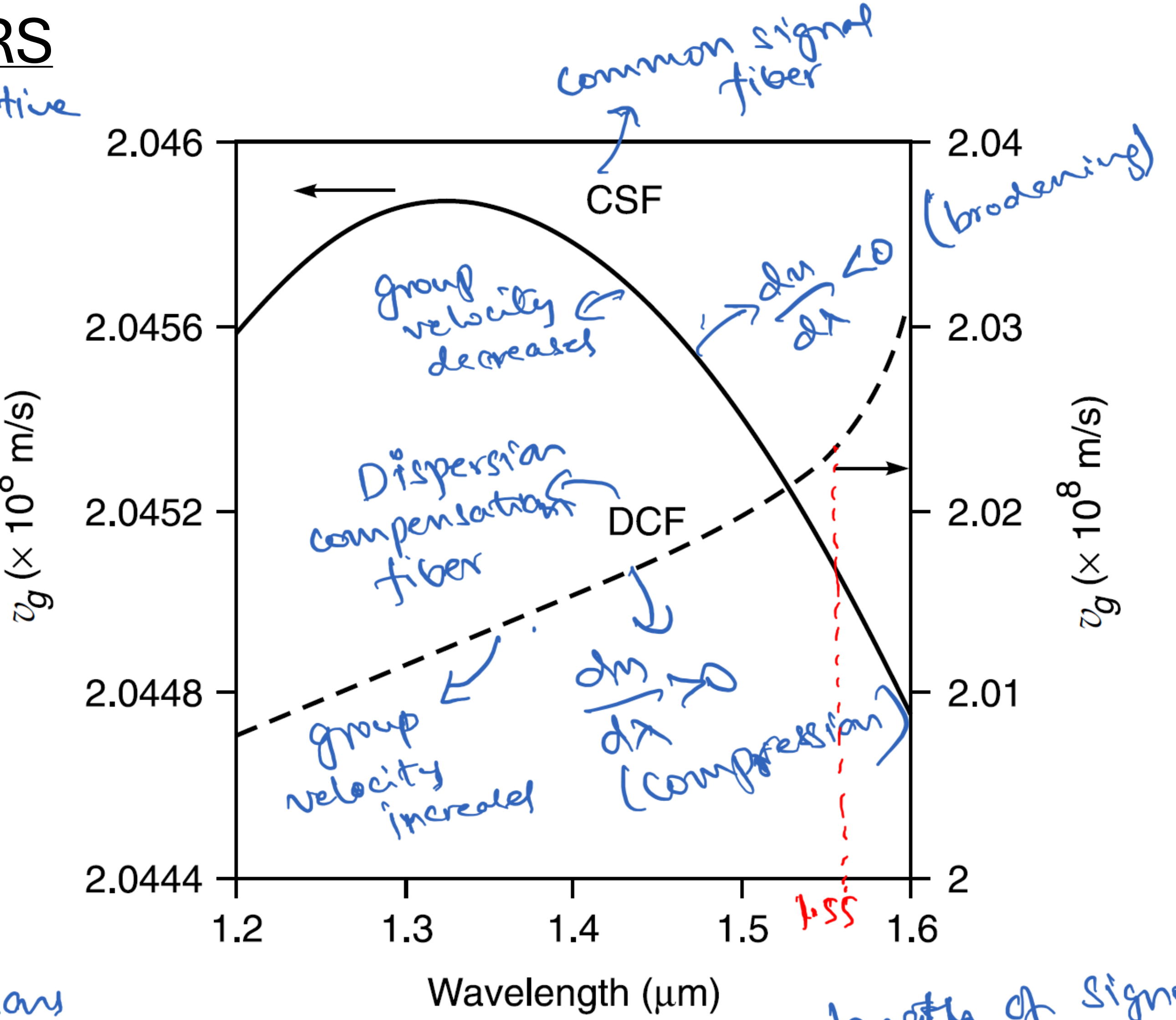


$$\boxed{D_1 L_1 + D_2 L_2 = 0}$$

$D_1 L_1$ + $D_2 L_2$ → No dispersion

single mode fiber (positive dispersion) | dispersion compensating fibers (negative dispersion) → No dispersions

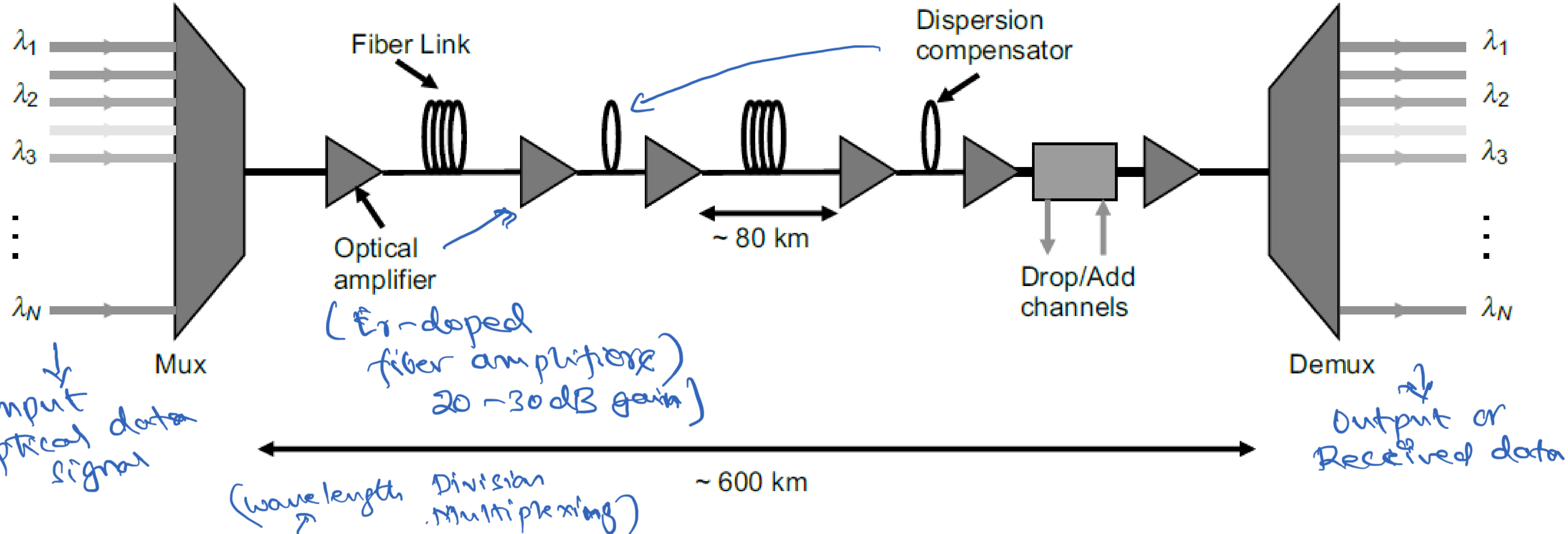
pulse broadening ($\Delta\tau = 2$ ps/km·nm) | pulse compression $\Delta\tau = -1800$ ps/km·nm → Retrieval of original pulse (signal)



$L_1 \sim 10$'s kms → length of signal fiber

$L_2 \sim 100$'s meters → length of dispersion compensating fiber

FIBER OPTIC COMMUNICATION SYSTEM



A typical WDM Fiber optic system with each wavelength carrying an independent challenl

$\lambda_1 - \lambda_N$ → ranges from 1530nm to 1560nm with the very low or sharp ($\Delta\lambda$) to be able to send high data signals.

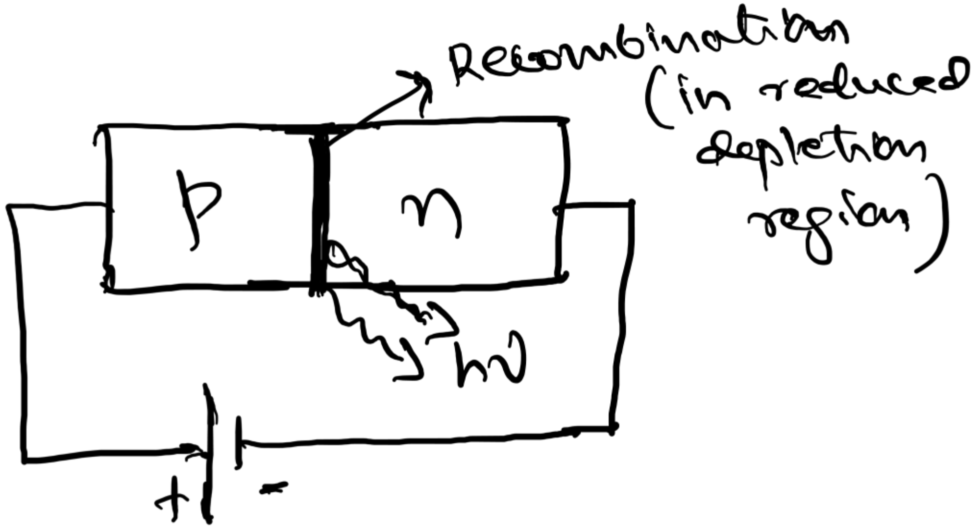
Semiconductor Laser

→ heterojunction diode

→ Forward bias → Recombination of e–h pair

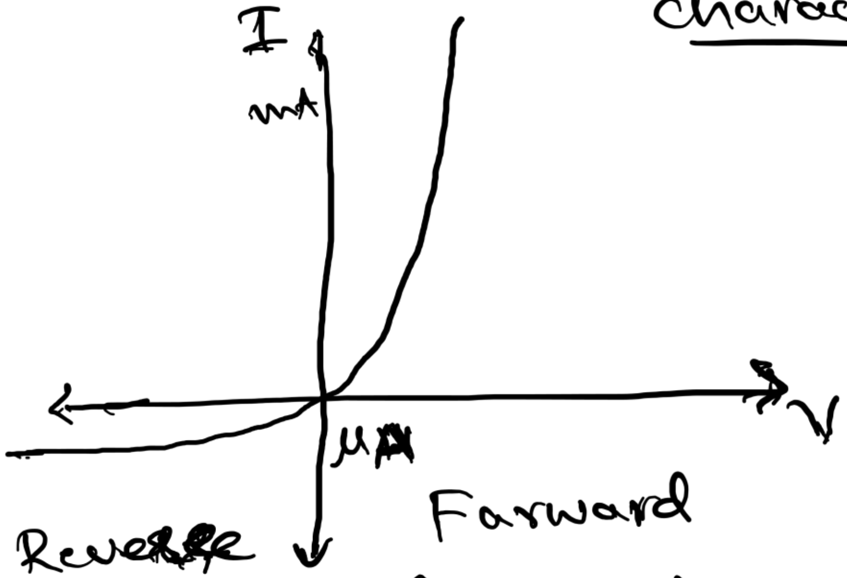
→ Gain spectrum

→ Feedback cavity (DFB)

→ 

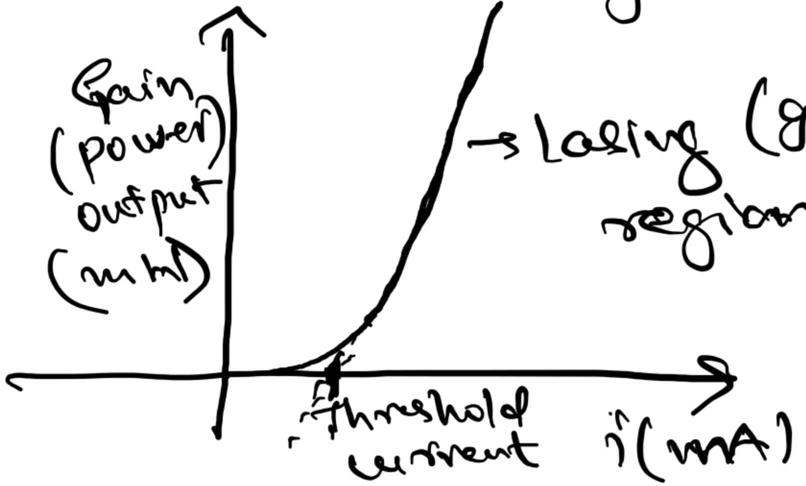
→ Direct bandgap semiconductors (GaAs, AlGaAs, InP) ..

diode I-V characteristics



Reverse (detector) (low current)

Forward (emitter) (high current gain)



PHOTODETECTORS

→ A device that measures the photon flux or optical power by converting the energy of the observed photon in to a measurable form (like voltage, temperature, ...)

Two classes

↳ ① Thermal detectors → converts the photon energy to heat.

Slow & low efficient detectors.

② Photoelectric detectors → Converting the photon energy directly to the electrical signal (current, voltage)

→ photo effect

→ Absorption of photon, creates e-h pair,

↳ ① External photoeffect → Photoelectric emission

Photo electric effect → photoelectrons escape from the material as free electrons (metal)

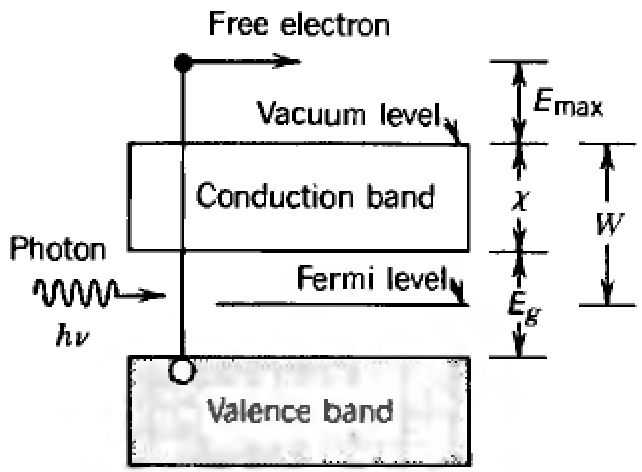
② Internal photoeffect → excited photo carriers remain with in the material (semiconductor)

Upon photo incidence conductivity of semiconductor increases

↑

Photoconductivity in semiconductor

External photoeffect



→ **Photoelectric effect**

$$KE_{max} = h\nu - h\nu_0$$

$$\Updownarrow$$

$$\boxed{\frac{1}{2} m v_{max}^2 = h\nu - W}$$

W → work function

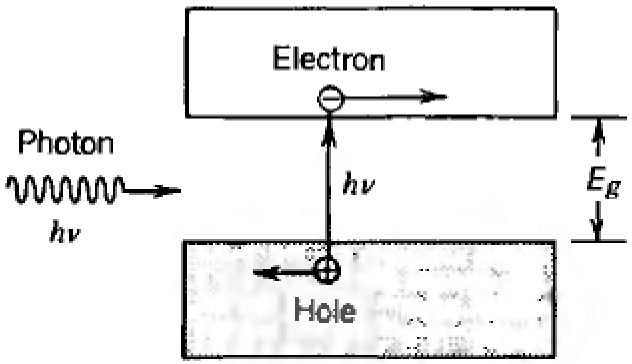
$h\nu$ → photon energy

KE_{max} → kinetic energy of free electron

Eg:- Photomultiplier tubes (PMTs)

Internal photoeffect

Eg:- → Photodiodes
→ Avalanche photo diodes (APDs)
→ photoconductor detectors



→ **Photo Conductivity**

→ Light induced increase in electrical conductivity of semiconductors

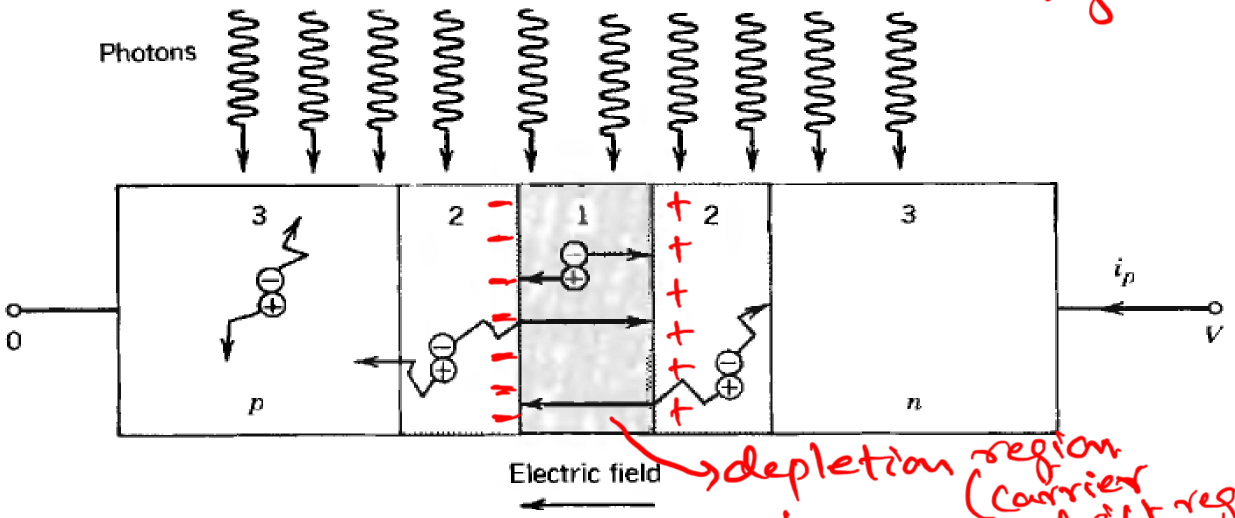
→ Absorption of photon results in generation of $\bar{e}$-hole pairs

→ Transport of $\bar{e}$-hole generates electrical current

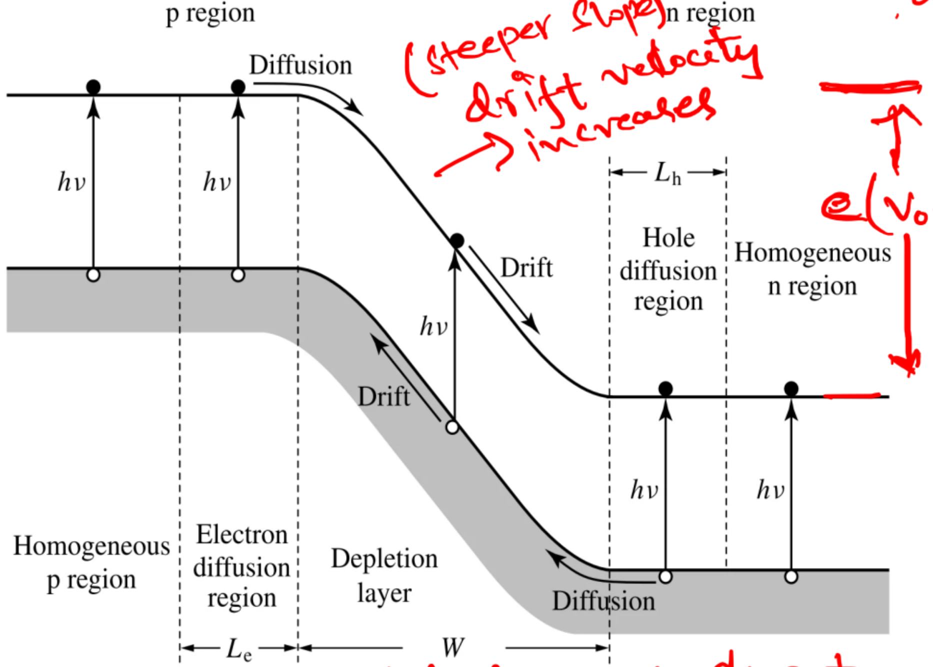
$$\boxed{\begin{array}{c} \text{Photon flux} \xrightarrow[\text{Q.E}]{} \text{current density} \\ \downarrow \\ \underset{\text{output}}{\text{voltage}} \xleftarrow[\text{gain}]{} \text{Amp} \leftarrow \text{charges} \end{array}}$$

Photodiode:

2 → diffusion region



→ depletion region (carrier drift region)

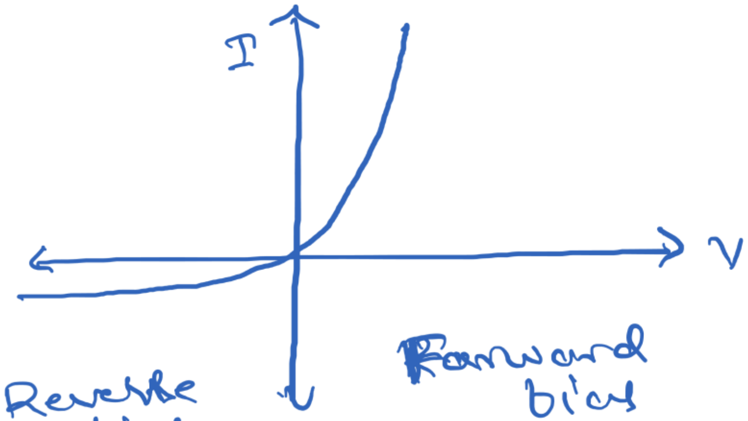


(steeper slope) drift velocity → increases

$e(V_0+V_B)$

Drift happens due to the potential difference

I-V characteristics



Reverse bias (Photodetector operation)

Forward bias (low resistance in the junction) (more conduction ⇓ Recombination)

→ In Reverse bias, the diode exhibits higher resistance (small reverse current) in the junction

$$I_D = I_0 \left[\exp\left(\frac{h\nu}{k_B T}\right) - 1 \right]$$

Diode current

I_0 → Reverse saturation current

→ The depletion region is much larger in Reverse bias vs Forward bias.

Advantageous of Reverse bias for photodetection

① The widening of depletion region improves the photon detection sensitivity due to large photon capture area.

② Reverse bias causes electrons to be pulled towards '+'ve terminal and holes to the '−'ve region, this reduces the transit time for electrons

⇒ higher mobility of charges ⇒ High speed or high frequency operation of photodetectors ($\sim 10\,GHz$)

③ Wider the depletion region ⇒ lower the internal capacitance ⇒ larger the cut-off frequency

$$C = \frac{\epsilon A}{d}$$

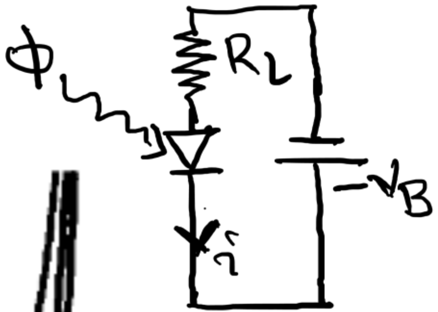
$$f = \frac{1}{2\pi RC}$$

$$\text{Transit time} = \frac{\text{depletion width}}{V_d}$$

$V_d \rightarrow$ drift voltage $= \mu E$, $\mu \rightarrow$ mobility of carriers

Higher $\mu \Rightarrow$ higher $V_d \Rightarrow$ lower transit time.

Reverse biased operation of photo diode :-

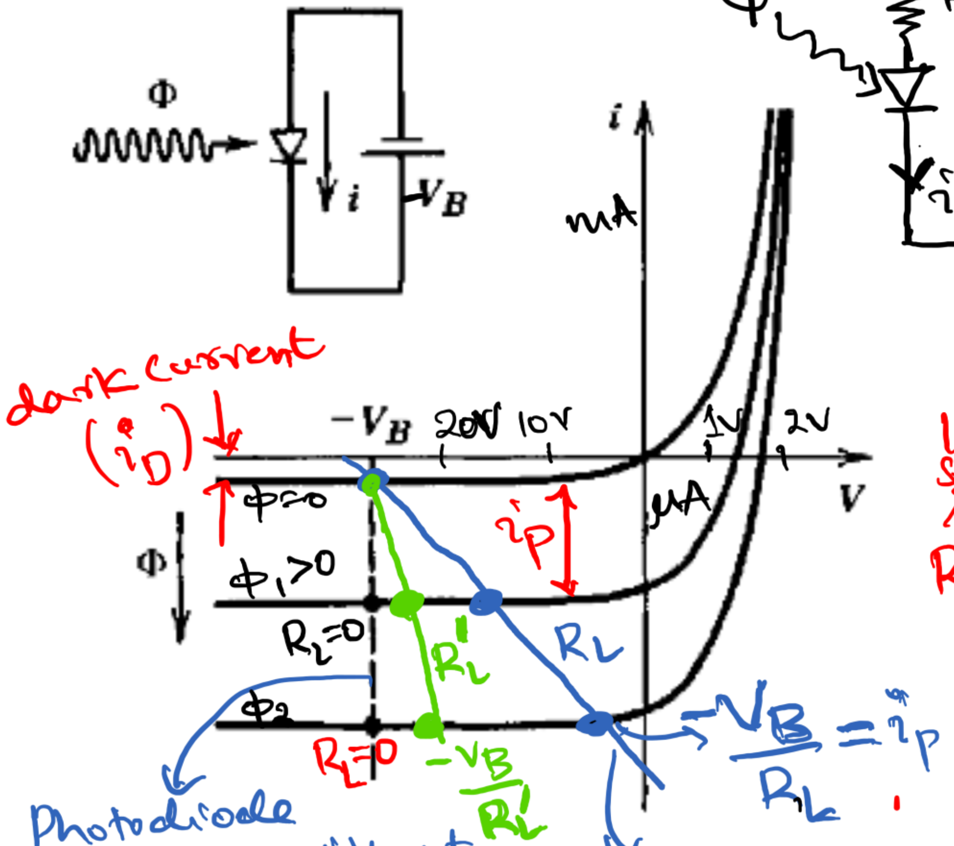


→ Photo conductive configuration

→ Dark current (i_D) is due to the thermally generated carriers in material in the absense of light

→ Larger the bandgap → lower the dark current

(Silicon → i_D → nA (optical frequency)
InGaAs → i_D → μA (IR - frequencies)
Ge → i_D → > μA (IR - frequencies)



low sensitivity

$R_L > R_L'$

higher sensitivity

Photodiode response without Load Resistance ($R_L = 0$)
⇓
No - saturation (Linear response) (ideal)

Photodiode response with Load resistance (R_L) (saturation of current (i_p) for higher ϕ (photon flux))

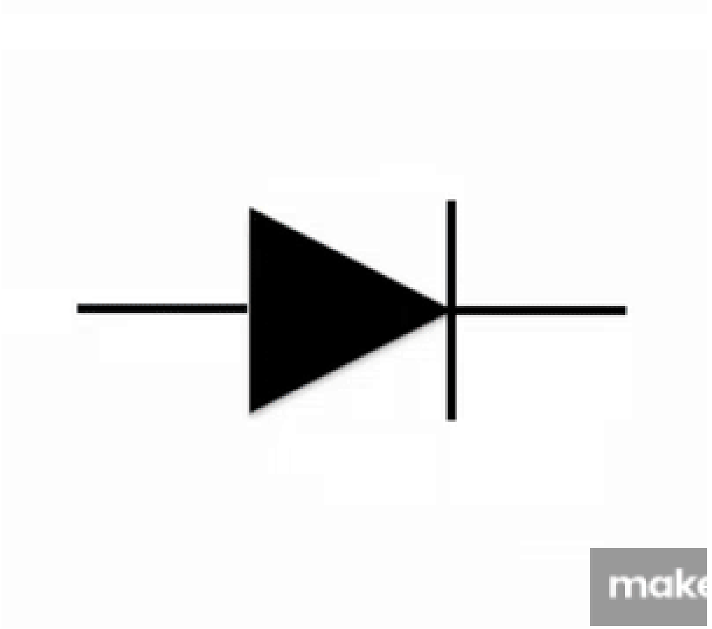
The diode current :-

$$I = I_0\left[\exp\left(\frac{h\nu}{kT}\right) - 1\right] - I_p$$

$I_0 \rightarrow i_D \rightarrow$ large $i_D \Rightarrow$ large noice (depends on temperature)

$I_p = i_p$

PIN-photodiode



Photodiode operation requires the creation of depletion region (intrinsic / charge neutral region)

→ No recombination

→ results in charge seperation

PIN diode

larger depletion region (intrinsic region)

↓

Larger the photo responsive area

↓

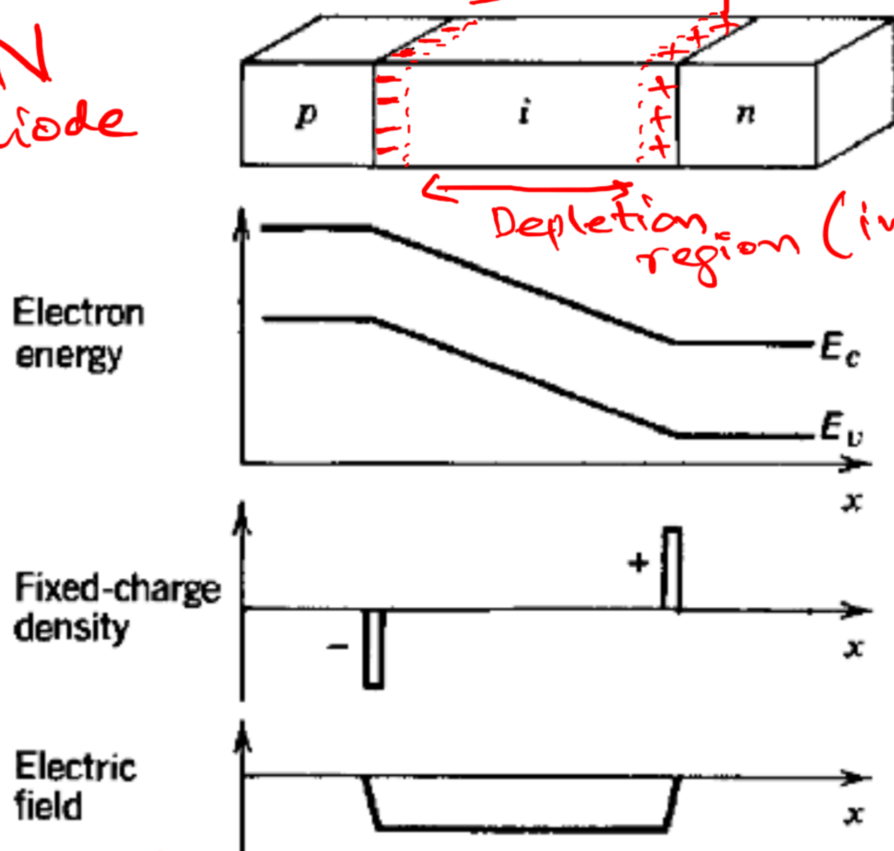
higher photon collection

↓

higher η

and

Smaller capacitance (higher cut-off frequencies)



No recombination of e⁻-holes. ⇐ constant Electric field in the depletion region

⇒ ~~the~~ e-hole pairs created by the photons will be seperated by this intrinsic electric field → later pushed away by the ~~external~~ external Reverse bias.

Properties of Semiconductor photodetectors → Performance indicators

① **Quantum efficiency (η)**:- Probability of converting the incident photon to electron-hole pairs in the semiconductor.

$$\eta = \frac{\text{number (flux) of electron-hole pair generated}}{\text{number (flux) of incident photons.}}$$

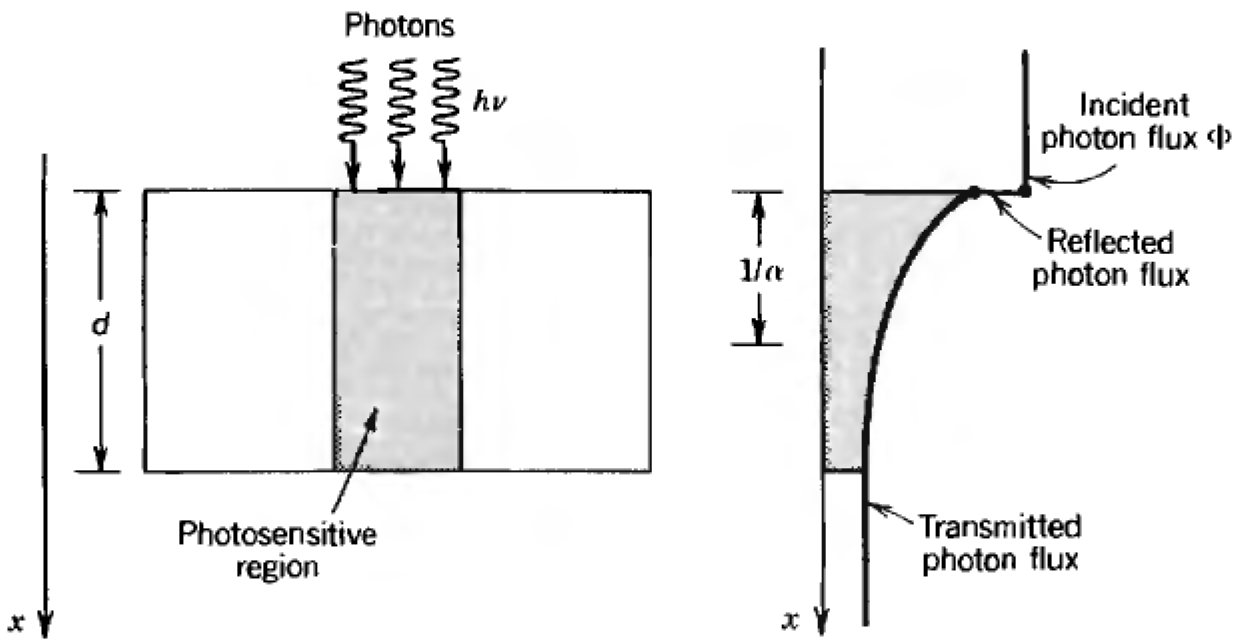
or
$$\eta = (1-R)\,\xi\,[1-\exp(-\alpha d)]$$

if $\alpha = 0$
$\Rightarrow \eta = 0$
$\Rightarrow$ absorption is must

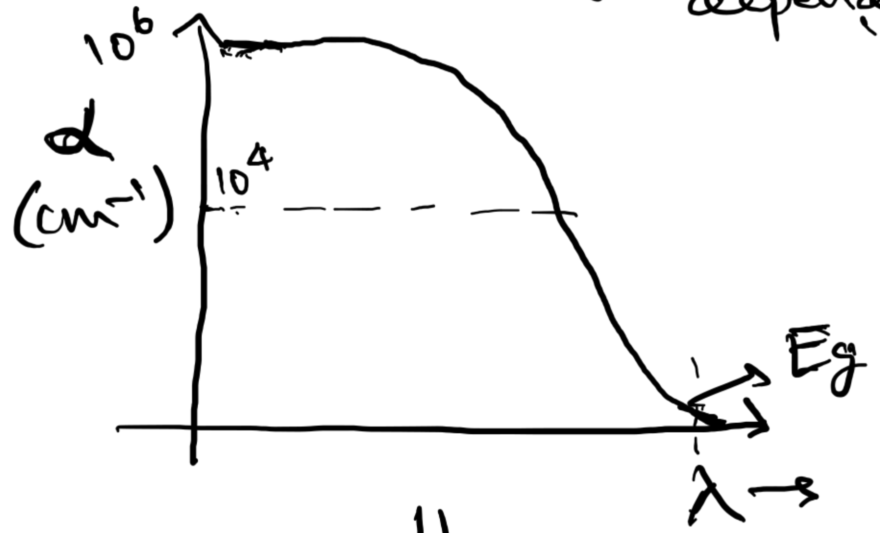
where, $R \to$ photon reflectance at the diode surface
$\xi \to$ fraction of electron-hole pairs contribute to the current
$\alpha \to$ absorption co-efficient of material (cm^{-1})
$d \to$ photodetector depth. $(cm) \to$ photosensitive region.

$$\boxed{0 \leq \eta \leq 1}$$

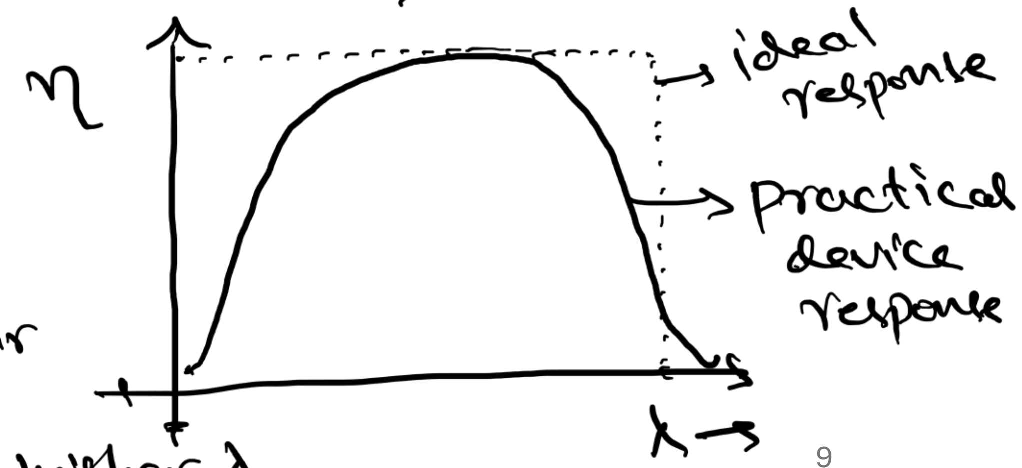
as 'α' increases
$\Rightarrow \eta$ increases
'R' reduces $\Rightarrow \eta$ increases
'd' increases $\Rightarrow \eta$ increases



Since absorption co-efficient 'α' depends on the wavelength (λ) of incident light ⇒ 'η' also depends on 'λ'



⇓



Effect of absorption (α & d) on η.

For example, $\alpha = 10^3\ cm^{-1}$

⇒ Penetration $1/\alpha = 10\ \mu m$

higher absorption for $\alpha = 10^6\ cm^{-1}$ ⇒ $1/\alpha = 1\ nm$

⇒ Most photons are absorbed near the detector surface

⇒ Photocurrent reduces ⇒ η decreases @ higher λ

② **Responsivity (R)** :- It relates the electric current flowing in the device to the incident optical power.

$$R = \frac{i_p}{P_{optical}} = \frac{\eta e}{h\nu} = \eta \frac{\lambda_0}{1.24} \qquad (A/W) \text{ unit}$$

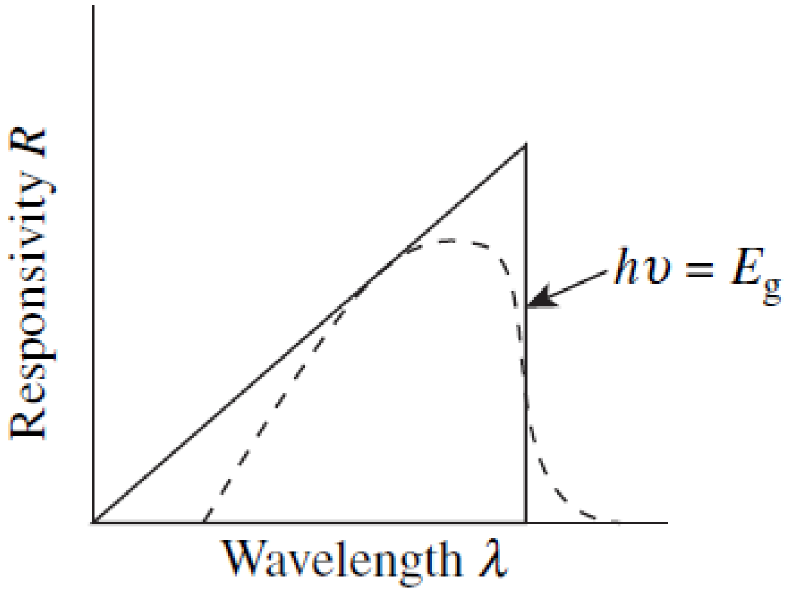
$i_p = \eta e \Phi$, Φ = photon flux

$P_{opt} = h\nu\Phi$

$i_p \rightarrow$ photoelectric current

$P_{opt} \rightarrow$ optical power.

$\Rightarrow \quad i_p \propto P_{opt} \quad \Rightarrow \quad i_p = R\, P_{opt}$

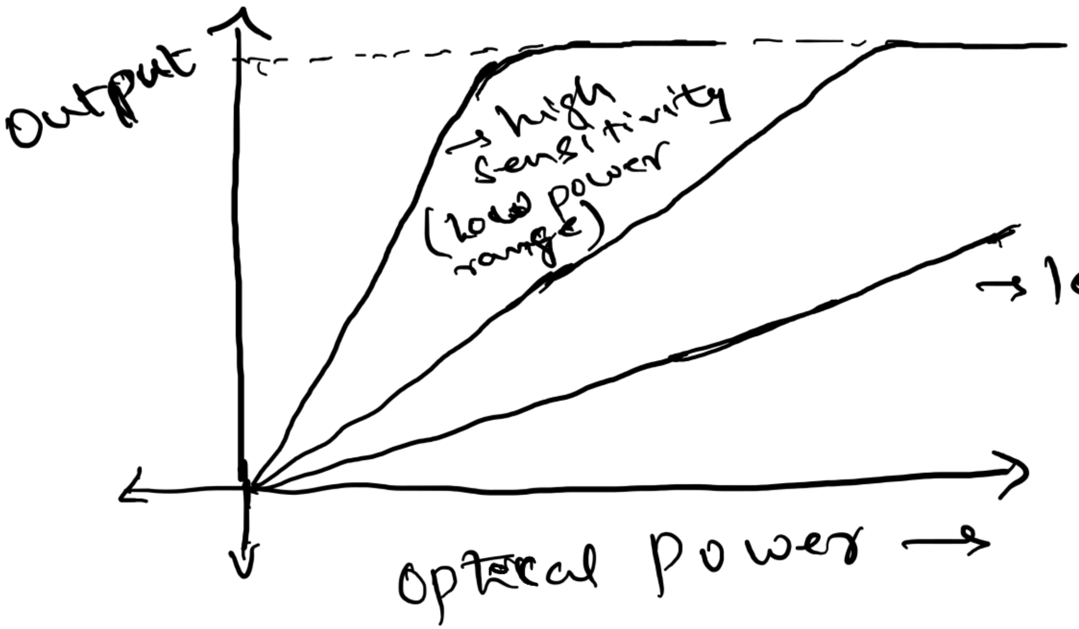


→ For a fixed 'η', R increases linearly with wavelength λ_0

→ The linearity in the Responsivity (R) with respect optical power → defines the dynamic range of the Photodetector

→ Above the dynamic range, the detector saturates.

③ <u>sensitivity</u> :- It describes the ability of Photodetector to convert incident light into a detectable electrical signal, quantified by Responsivity (R) and Detectivity (D) of the detector.



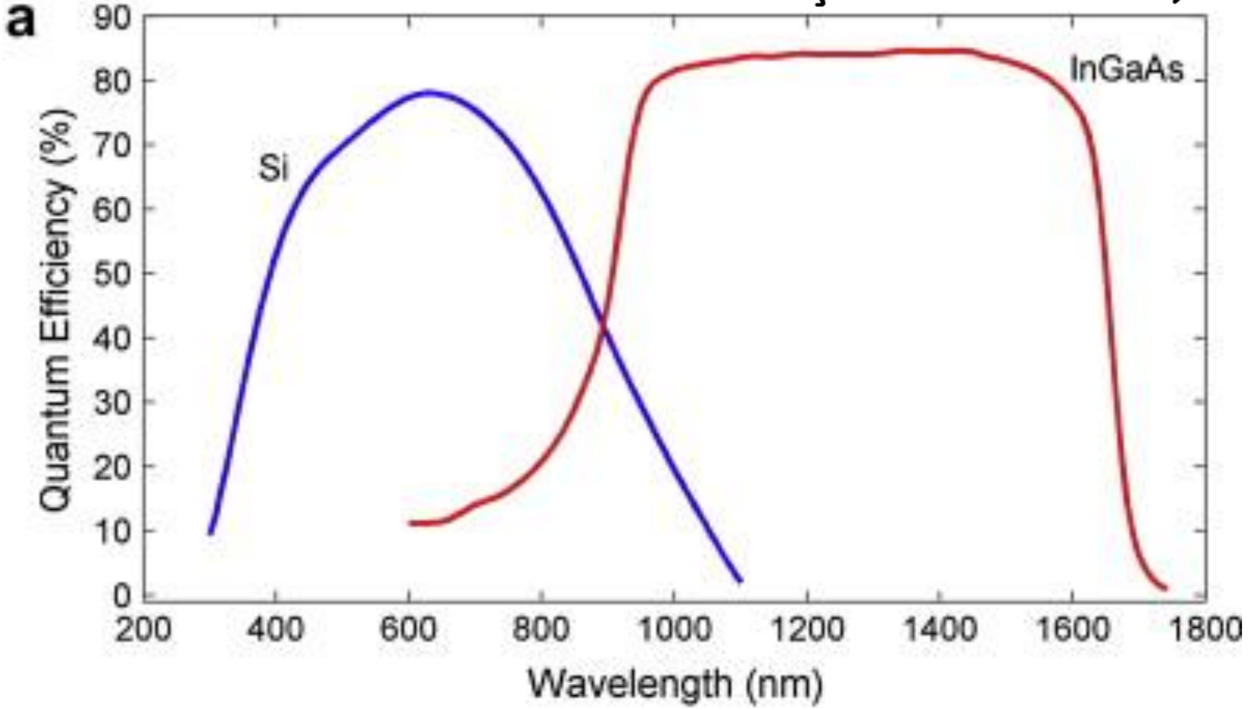
→ high sensitivity detectors will have low dynamic range
($\sim 10^{-6}$ W to 10^{-3} W)
µW to mW
dynamic range → 30 dB

→ less sensitive
(large power range)
(large linearity)
(large dynamic range)

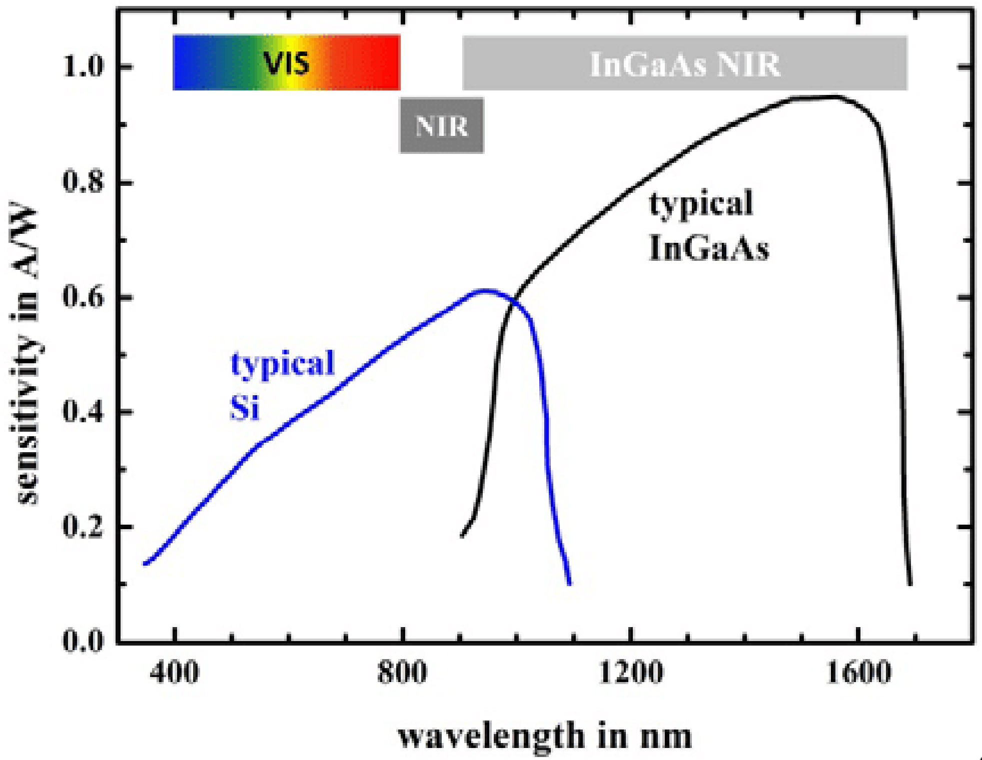
⇓

If the detector has <u>60 dB dynamic range</u> ⇒ the power measurement range 10^{-6} W to 1 W
10 µW to 1 W

Quantum efficiency (η)



Responsivity (A/W)



④ Response time (Rise time) :- Delay b/w the generation and collection of e^- in the external circuit. → also known as transit time spread.

$$i(t) = -\frac{Q}{W} v(t)$$

w → width of junction, Q → charge of carrier,

Noise in Photodetectors:

Sources of noises:-

(1) Dark current :- Due to thermally excited carriers, the photodetector has small current in the absense of photons

→ This limits the accuracy of measurements at low light levels.

(2) Shot noise :- The intrinsic fluctuations in the input light creates electrical noise in the external circuit

→ called shot noise. → It is a random noise.

→ associated with the poisson statistics.

(3) photoelectron noise :- Due to the inherent randomness in the process of carrier generation, where the $\eta < 1$

(4) Gain noise :- Amplification process provides internal gain that may be random for each photoelectron detection.

(5) Receiver circuit noise :- Electrical noise from the external circuit.

Performance indicators for sensitive detection :-

→ Signal to noise ratio (S/N) :- SNR is a measure of minimum detectable signal, is defined as

$$SNR = \frac{(\text{mean})^2}{\text{variance}} = \frac{(\bar{i})^2}{\sigma_i^2}$$

$i \rightarrow$ photocurrent

→ Minimum detectable signal, $SNR = 1$

→ SNR for photon noise $= \bar{n}$

$n \rightarrow$ number of photons

⇒ minimum detectable limit for photon number $\bar{n} = 1$ photon.

$\eta \rightarrow$ Quantum efficiency,

→ SNR for photoelectron noise $= \eta \bar{n}$

⇒ minimum detectable limit $\eta \bar{n} = 1$

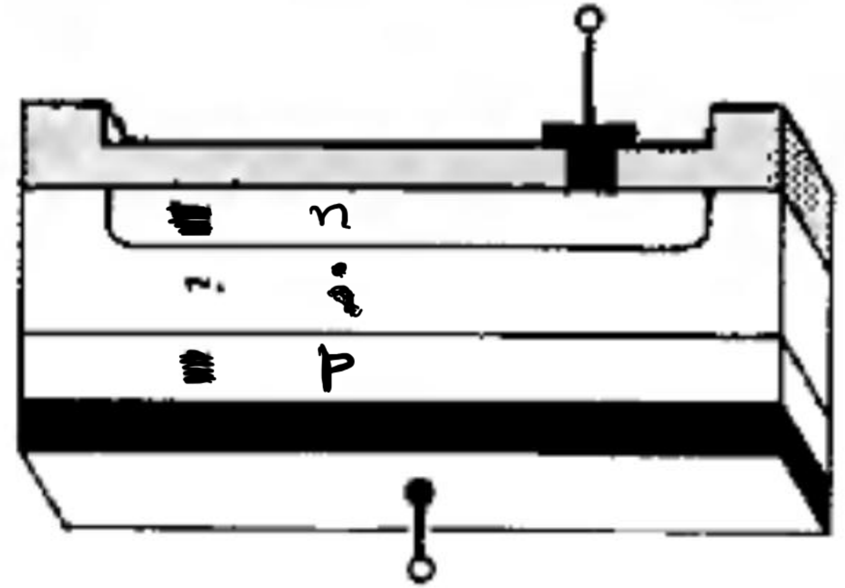
or $\bar{n} = 1/\eta$ photons.

→ SNR for photocurrent noise $= \frac{\eta \phi}{2B} = \bar{m}$,

$\phi \rightarrow$ photon flux

$B \rightarrow$ Electrical Bandwidth of circuit

p-i-n Photodiode



CHAPTER 1

Light Waves

1.1 INTRODUCTION

Visible light constitutes a small, albeit an important, segment of the broad spectrum of electromagnetic waves encompassing γ-rays on one extreme and radio waves on the other. Between these two extremes, lie X-rays, ultraviolet radiation, visible light, infrared radiation and microwaves in decreasing order of frequency (Table 1.1). At the present stage of development of the field of optics, it is really not necessary to justify the wave nature of light. Having said that, it must also be mentioned that the original controversy between the two protagonists (Sir Issac Newton and Christian Huygens) representing two schools of thought – light being corpuscular and light having wave nature – took a new twist with the development of quantum mechanics. Light, like matter, is now understood to have a dual character – the wave-like behavior as well as the particle-like (photon) behavior. Both attributes may not be revealed in a single measurement. Broadly speaking, light propagation in free space and in other media can be described in classical terms whereas light–matter interaction (absorption and emission of light) can be understood only in the quantum mechanical description. In this book, we are primarily concerned with light propagation and hence the classical description in terms of Maxwell's equations is quite adequate. Maxwell's equations predict the velocity of propagation of electromagnetic waves in vacuum which is in close agreement with the measured velocity of light. This observation firmly establishes light in the realm of the electromagnetic waves.

1.2 MAXWELL'S EQUATIONS

All electromagnetic phenomena, including light propagation, can be fully described in terms of Maxwell's equations (written here, in the SI units):

$$\nabla \cdot \vec{E} = \rho/\epsilon_0,$$

$$\nabla \cdot \vec{B} = 0,$$

Table 1.1. The electromagnetic spectrum.

Spectral Region	Approximate Frequency Range
Gamma rays	$>10^{20}$ Hz
X-rays	$10^{17}-10^{20}$ Hz
Ultraviolet	$10^{15}-10^{17}$ Hz
Visible	$(3.5-7.5)\times 10^{14}$ Hz
Infrared	$10^{12}-10^{14}$ Hz
Microwaves	$10^{9}-10^{12}$ Hz
Radiofrequency	$<10^{9}$ Hz

$$\nabla\times \vec{E} = -\frac{\partial \vec{B}}{\partial t},$$

$$\nabla\times \vec{B} = \mu_0\left(\vec{J} + \epsilon_0\frac{\partial \vec{E}}{\partial t}\right), \tag{1.1}$$

where μ_0 and ϵ_0 are, respectively, the permeability and permittivity of vacuum; ρ and $\vec{J}$ are the charge and current densities, respectively.

There is a need to distinguish between the microscopic and macroscopic forms of Maxwell's equations. The charge and current densities in the microscopic form of Maxwell's equations are those which exist at the atomic level. Consequently, the electric field $\vec{E}$ and magnetic field $\vec{B}$ are expected to show rapid variations over atomic and subatomic distances. Visible light with wavelength range between 400 and 800 nm cannot probe the charge and current distributions at the atomic level. X-rays and γ-rays with much shorter wavelengths are better suited to probe atomic distributions. Light waves can provide information on charge and current distributions in matter averaged over distances of the order of the wavelength of light. In that sense, light is a rather crude probe to interrogate matter at the atomic level. Light waves perceive a medium more like a continuum, and not a medium packed with discrete particles. The macroscopic form of Maxwell's equations uses the charge and current densities which are averaged over microscopically large, but macroscopically small volumes. Macroscopically averaged fields vary smoothly in space and are mathematically well behaved. The Gauss and Stokes vector theorems can be applied to these fields. In this book, we shall deal with the macroscopic form of Maxwell's equations. Maxwell's equations in the differential form (Eq. 1.1) can be derived from the empirical integral formulation of the laws of electromagnetism developed over centuries by Gauss, Ampère, Faraday and others. Maxwell brought symmetry to these equations by introducing the displacement current density $\epsilon_0\partial\vec{E}/\partial t$. No wonder, these equations are known as Maxwell's equations. In the context of the macroscopic form of Maxwell's equations, it is necessary to distinguish between the free and bound charge and current densities. The free electrons in conductors generate the free charge density (ρ_f). In addition, it may also happen that the centers of the positive and negative charges in a small macroscopic volume may not coincide. If this happens, an electric dipole moment can be associated with this volume and the medium is said to be polarized. The electric polarization $\vec{P}$ is defined as

$$\vec{P} = \frac{\text{net electric dipole moment in a macroscopically small volume } V}{\text{volume } V}. \quad (1.2)$$

The bound charge density in a polarized medium is given by

$$\rho_b = -\nabla \cdot \vec{P}. \quad (1.3)$$

The bound charge density ρ_b is non-zero only if polarization $\vec{P}$ is spatially changing. Electric polarization can be created in a medium either by aligning its polar molecules or by displacing its negative charge with respect to the positive charge by the application of an external electric field. The movement of the free charges in a conductor gives rise to the free current density ($\vec{J}_f$), and the changing displacements of the bound charges from their equilibrium positions give rise to the bound current density

$$\vec{J}_b = \frac{d\vec{P}}{dt}.$$

We should also recognize the existence of the magnetic dipole moments in magnetic materials. The bound current density can be generalized to include these contributions as well;

$$\vec{J}_b = \frac{d\vec{P}}{dt} + \nabla \times \vec{M}, \quad (1.4)$$

where magnetization $\vec{M}$ is the magnetic moment per unit volume defined in the manner of Eq. (1.2). We now write Maxwell's equations indicating these contributions explicitly:

$$\nabla \cdot \vec{E} = (\rho_f + \rho_b)/\epsilon_0, \quad (1.5a)$$

$$\nabla \cdot \vec{B} = 0, \quad (1.5b)$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}, \tag{1.5c}$$

$$\nabla \times \vec{B} = \mu_0 \left(\vec{J}_\mathrm{f} + \vec{J}_\mathrm{b} + \epsilon_0 \frac{\partial \vec{E}}{\partial t} \right). \tag{1.5d}$$

These equations along with the defining equations for the bound charge and bound current densities constitute a formidable set of equations to deal with. They can be made more compact by introducing two additional fields,

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P}, \tag{1.6a}$$

$$\vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M}, \tag{1.6b}$$

where $\vec{D}$ is the electric displacement field and $\vec{H}$ is the magnetic field. The field $\vec{B}$ is usually called the magnetic induction or the magnetic flux density. The term magnetic field is often used to refer either of the $\vec{B}$ or $\vec{H}$ field. Maxwell's equations (Eqs 1.5) can now be put in the form:

$$\nabla \cdot \vec{D} = \rho_\mathrm{f}, \tag{1.7a}$$

$$\nabla \cdot \vec{B} = 0, \tag{1.7b}$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}, \tag{1.7c}$$

$$\nabla \times \vec{H} = \vec{J}_\mathrm{f} + \frac{\partial \vec{D}}{\partial t}. \tag{1.7d}$$

Despite the presence of the source terms, Maxwell's equations should not be conceptualized in terms of the cause and effect, where the fields are determined by the sources of the charge and current present in the medium. The sources and fields are, in fact, inter-dependent – each affecting the other. True, the free charges do not depend on the fields, but the bound charges and currents are field dependent. The bound charges and currents change the fields and are in turn modified by the changing fields.

Equations (1.7) appear deceptively simple but are actually unmanageable primarily because, notwithstanding Eqs (1.6), no simple relationships exist between the electric fields $\vec{E}$ and $\vec{D}$ and between the magnetic fields $\vec{B}$ and $\vec{H}$. Fortunately, the elementary magnetic moments are not of much concern at the optical frequencies. Consequently, the magnetization $\vec{M}$ can be ignored and the relationship between the $\vec{B}$ and $\vec{H}$ fields for materials of optical interest is rather simple:

$$\vec{B} = \mu \vec{H}.$$

The permeability μ of optical materials is essentially field independent and differs only slightly from vacuum permeability μ_0. However, the electric polarization $\vec{P}$ must be reckoned with and cannot be ignored. In the absence of a detailed understanding in classical terms, the electric polarization $\vec{P}$ is usually expanded as a power series in the electric field:

$$P_i = \epsilon_0 \left[\chi_{ij}^{(1)} E_j + \chi_{ijk}^{(2)} E_j E_k + \chi_{ijkl}^{(3)} E_j E_k E_l + \cdots \right], \tag{1.8}$$

where E_i, E_j, E_k are the components of the electric field contributing to the ith component of the polarization $\vec{P}$. The coefficients $\chi^{(n)}$ with $n = 1, 2, 3, \ldots$ are the electric susceptibility tensors describing intrinsic material properties and are best understood in quantum mechanical terms. Alternatively, they may be treated as parameters to be determined empirically. Equation (1.8) is actually more complicated than it appears because the polarization $\vec{P}$ at a certain space-time point $(\vec{r}, t)$ may depend, in addition to field $\vec{E}$ at point $\vec{r}$ and time t, on fields in the spatial neighborhood of this point and may also depend on fields at times prior to the chosen time t. We shall ignore such complications. Here, we assume polarization $\vec{P}(\vec{r}, t)$ to depend linearly on the local and instantaneous field only. Hence, we can write

$$\vec{P}(\vec{r}, t) = \epsilon_0 \chi^{(1)} \vec{E}(\vec{r}, t). \tag{1.9a}$$

This is the regime of linear optics to which most of this book is devoted. The remaining terms in Eq. (1.8) form the basis of the exciting field of nonlinear optics (Chapter 14). Equation (1.9a) is equivalent to

$$\vec{D}(\vec{r}, t) = \epsilon \vec{E}(\vec{r}, t), \tag{1.9b}$$

where

$$\epsilon = \epsilon_0 (1 + \chi^{(1)}) \tag{1.9c}$$

is the medium permittivity. Except for vacuum ($\chi^{(1)} = 0$), the linear susceptibility $\chi^{(1)}$ and permittivity ϵ are in general complex suggesting the polarization $\vec{P}$ and displacement field $\vec{D}$ do not always remain in phase with the electric field $\vec{E}$. For conducting media, the so-called constitutive relations (Eqs 1.6) need to be supplemented by

$$\vec{J} = \sigma \vec{E}, \tag{1.6c}$$

where σ is the electrical conductivity of the medium. A homogeneous medium is characterized by constant values of ϵ, μ and σ, and an inhomogeneous medium admits changes in these quantities from point to point in a smooth manner. For linear optical materials ($\rho_f = 0$, $\vec{J}_f = 0$, $\sigma = 0$), Eqs (1.7) can be re-cast into the form:

$$\nabla \cdot \epsilon \vec{E} = 0, \tag{1.10a}$$

$$\nabla \cdot \vec{B} = 0, \tag{1.10b}$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}, \tag{1.10c}$$

$$\nabla \times \vec{B} = \mu\epsilon \frac{\partial \vec{E}}{\partial t}. \tag{1.10d}$$

We note that for linear optical materials, only two fields $\vec{E}$ and $\vec{B}$ need to be dealt with, but the permittivity ϵ and to some extent the permeability μ are unknown quantities to be determined with reference to experimental observations. It must be appreciated that the averaging process has transferred the information on the electromagnetic behavior of the medium at the atomic level to the macroscopic or bulk properties of the medium – the permittivity ϵ and permeability μ in the context of optical materials.

All electromagnetic fields including the light fields must be consistent with Maxwell's equations, but on their own these equations do not suggest the existence of fields of any particular kind. One needs to postulate specific forms of the fields and then obtain conditions for their existence. Another point to be noted is that these equations describe relationships for the spatial and temporal variations of the fields, but do not provide any clue as to how these fields are generated in the first place.

1.3 THE WAVE EQUATION

The electric and magnetic fields appear coupled in Maxwell's equations. It is possible to de-couple them. The decoupling process brings out some of the most exciting aspects of electromagnetism. For a homogeneous medium, except at its boundaries, Eq. (1.10a) reduces to

$$\nabla \cdot \vec{E} = 0. \tag{1.10e}$$

This result in conjunction with Eq. (1.5a) suggests that a linear homogeneous medium, with no free charge inside, cannot sustain any bound charge except (may be) at its boundaries. We shall have to fall back to Eq. (1.10a) when the boundaries of a homogeneous medium are approached. With Eq. (1.10e), the $\nabla \times \nabla \times \vec{E}$ simplifies to

$$\nabla \times \nabla \times \vec{E} = \nabla(\nabla \cdot \vec{E}) - \nabla^2 \vec{E} = -\nabla^2 \vec{E}\,.$$

Taking curl of Eq. (1.10c), interchanging ∇ and $\partial/\partial t$ operations on the right-hand side and combining it with Eq. (1.10d) leads to the well-known wave equation

$$\nabla^2 \vec{E} - \mu\epsilon \frac{\partial^2 \vec{E}}{\partial t^2} = 0. \tag{1.11a}$$

In a similar manner, we can obtain

$$\nabla^2 \vec{B} - \mu\epsilon \frac{\partial^2 \vec{B}}{\partial t^2} = 0. \tag{1.11b}$$

Notwithstanding this apparent separation, the electric field $\vec{E}$ and magnetic field $\vec{B}$ of an electromagnetic wave remain dependent on each other through Maxwell's equations.

The wave equations (1.11) describe wave motion in a variety of situations, as for example the waves in an elastic medium. We can interpret Eqs (1.11) to describe the propagation of the electric and magnetic fields or more appropriately, the propagation of the electromagnetic waves. Extending the similarity with the elastic waves a bit further, one may postulate the existence of some kind of an elastic medium pervading all space which makes it possible for the electromagnetic waves to propagate. Aether was thought to be such a medium. It must necessarily be a thin medium since electromagnetic waves do propagate in essentially free space. At the same time, aether must be sufficiently elastic for wave propagation to take place. These are some of the internal inconsistencies of the aether postulate. The results of an ingenious experiment performed by Michelson and Morley were not consistent with the aether postulate. Aether has no place in the special theory of relativity developed by Albert Einstein.

Electromagnetic waves including the light waves can propagate in absolutely empty space. They do not require matter to facilitate propagation. The changing electric and magnetic fields associated with an electromagnetic wave are capable of sustaining each other. A comparison of the wave equation with its counterpart for mechanical waves suggests that the product $\mu\epsilon$ must represent the inverse of the square of the speed of propagation of electromagnetic waves. A medium is not necessary for the propagation of electromagnetic waves. However, the velocity of propagation of electromagnetic waves in a given medium is determined by its permeability and permittivity. The vacuum with permeability $\mu_0 = 4\pi \times 10^{-7}\,\mathrm{N\,s^2\,C^{-2}}$ and permittivity $\epsilon_0 = 8.85 \times 10^{-12}\,\mathrm{C^2\,N^{-1}\,m^{-2}}$ has velocity $c = 2.99 \times 10^8\,\mathrm{m\,s^{-1}}$ for the propagation of electromagnetic waves. This value agrees very closely with the velocity of light measured in the laboratory. This brings light within the domain of applicability of Maxwell's equations.

The wave equation (1.11) is a linear, homogeneous, second-order differential equation. The linearity of the wave equation leads to the superposition principle which states that if $\vec{E}_j$ $(j = 1, 2, 3, \ldots, n)$ are solutions of the wave equation, then $\sum_j a_j \vec{E}_j$ is also a solution of the wave equation, where a_j are arbitrary constants (real or complex). The wave equation admits a variety of solutions – some extremely simple in form, others sufficiently intricate. The implication of this statement needs to be appreciated. All light fields in a homogeneous medium must be solutions of the wave equation. However, external conditions must be accurately controlled to generate light fields to correspond to a particular solution of the wave equation. Some solutions may be mathematically easy to handle, but difficult to realize in practice. Fortunately, external conditions can often be manipulated to favor a particular kind of solution – generation of coherent light in a laser is an important step in this direction. The plane wave solution

$$\vec{E}(\vec{r}, t) = \vec{E}_0\, \mathrm{e}^{\mathrm{i}(\vec{k}.\vec{r} - \omega t)}$$

is perhaps the simplest solution and the lowest order Bessel wave solution [1.2]

$$E(\vec{r}, t) = E_0 J_0(\alpha\rho)\mathrm{e}^{\mathrm{i}(\beta z - \omega t)},$$

representing a nonspreading beam with $\alpha^2 + \beta^2 = (\omega/c)^2$, is one of the non-trivial solutions of the wave equation.

A plane wave is actually unphysical in the sense that no experimental effort can succeed to generate a plane wave. Notwithstanding this 'awkwardness', the plane wave solution of the wave equation is an extremely useful solution. In the backdrop of these remarks, we now discuss some monochromatic (single frequency) solutions of the wave equation in a homogeneous medium. The quasi-monochromatic and polychromatic wave solutions can be constructed in terms of the monochromatic wave solutions. This will be the subject matter of the next chapter.

1.3.1 Plane Wave Solution

The general solution of the wave equation (1.11) can be written in the form

$$\vec{E}\,(\vec{r}, t) = \vec{E}_0\,(\vec{r}, t)\mathrm{e}^{\mathrm{i}\phi(\vec{r}, t)}, \tag{1.12}$$

where $\vec{E}_0\,(\vec{r}, t)$ and $\phi(\vec{r}, t)$ are the amplitude and phase of the wave, respectively. A plane wave is characterized by phase $\phi(\vec{r}, t)$ which, at any given time, remains constant in a plane perpendicular to the direction of propagation of the wave. The phase

$$\phi(\vec{r}, t) = \vec{k} \cdot \vec{r} - \omega t$$

satisfies this condition since the dot product $\vec{k} \,.\, \vec{r}$ remains constant $(= kr_0)$ as the tip of the position vector $\vec{r}$ moves over a given plane perpendicular to the direction of propagation $\vec{k}$; r_0 is the component of $\vec{r}$ in the direction of $\vec{k}$ (Fig. 1.1). The amplitude $\vec{E}_0$ of a plane wave does not depend on position vector $\vec{r}$ and time t.

A surface (in this case a plane) of constant phase is called a wavefront or an equiphase surface. Let plane I in Fig. 1.1 represent the wavefront at the space-time point (r_0, t_0) with phase

$$\phi_0 = \vec{k} \cdot \vec{r} - \omega t = kr_0 - \omega t_0.$$

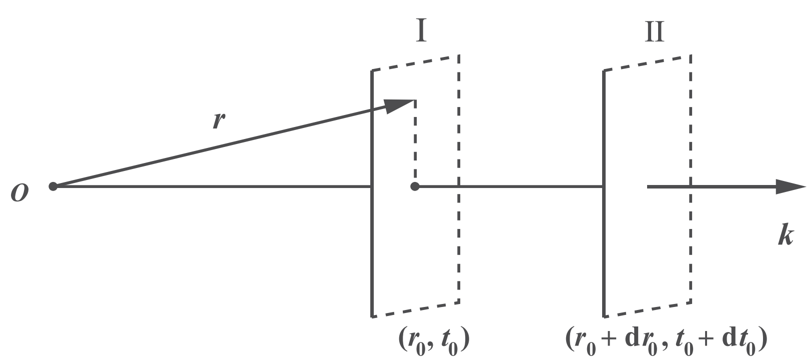


Fig. 1.1: Moving wavefront of a plane wave.

This wavefront moves along with the wave and plane II is its subsequent position at the neighboring space-time point $(r_0 + \mathrm{d}r_0, t_0 + \mathrm{d}t_0)$. Therefore,

$$\phi_0 = kr_0 - \omega t_0 = k(r_0 + \mathrm{d}r_0) - \omega(t_0 + \mathrm{d}t_0).$$

The velocity of propagation of the wavefront is given by

$$v_\mathrm{p} = \frac{\mathrm{d}r_0}{\mathrm{d}t_0} = \frac{\omega}{k}.$$

This is the phase velocity or the wave velocity. We could have defined the phase of a plane wave with a negative sign before $\vec{k} \cdot \vec{r}$. That choice represents another plane wave propagating in just the opposite direction. In fact, any well-behaved mathematical function of $(\pm k \cdot r - \omega t)$ can represent a plane wave. A particularly useful form of the plane wave is the harmonic plane wave

$$\vec{E}_\mathrm{r} = \vec{E}_{0\mathrm{r}} \cos(\vec{k} \cdot \vec{r} - \omega t + \phi_0) \tag{1.13a}$$

in the real field notation or

$$\vec{E} = \vec{E}_{0\mathrm{r}}\, \mathrm{e}^{\mathrm{i}(\vec{k} \cdot \vec{r} - \omega t + \phi_0)} \tag{1.13b}$$

in the complex field notation, where ϕ_0 is a constant called the phase constant. To avoid trigonometric complications, we prefer to employ the complex field notation. The real field can always be recovered from the complex field and its complex conjugate:

$$\vec{E}_\mathrm{r} = \frac{1}{2}\, \vec{E}_{0\mathrm{r}}\, \mathrm{e}^{\mathrm{i}(k \cdot r - \omega t + \phi_0)} + \frac{1}{2}\, \vec{E}_{0\mathrm{r}}\, \mathrm{e}^{-\mathrm{i}(\vec{k} \cdot \vec{r} - \omega t + \phi_0)}. \tag{1.14}$$

A more general harmonic plane wave is described by the fields

$$\vec{E} = \vec{E}_0\, \mathrm{e}^{\mathrm{i}(\vec{\tilde{k}} \cdot \vec{r} - \omega t)}, \tag{1.15a}$$

$$\vec{B} = \vec{B}_0\, \mathrm{e}^{\mathrm{i}(\vec{\tilde{k}} \cdot \vec{r} - \omega t)}. \tag{1.15b}$$

The notation $\vec{\tilde{k}}$ is used to distinguish the complex wave vector from the real wave vector $\vec{k}$. The complex wave vector or the propagation vector $\vec{\tilde{k}}$ allows for the attenuation (or the gain) of the amplitude of a wave as it propagates in the medium. For complex $\vec{E}_0$ and $\vec{B}_0$, the electric and magnetic fields may not always remain in phase. The complex propagation vector may be expressed as

$$\vec{\tilde{k}} = \vec{k} + \mathrm{i}\,\vec{a}, \tag{1.16}$$

where $\vec{k}$ is the real part of the propagation vector and $\vec{a}$ is a real vector called the attenuation vector. For the harmonic plane wave solution to be consistent with Maxwell's equations in a homogeneous medium, following conditions must be satisfied:

$$\vec{\tilde{k}} \cdot \vec{E}_0 = 0, \tag{1.17a}$$

$$\vec{\tilde{k}} \cdot \vec{B}_0 = 0, \tag{1.17b}$$

$$\vec{B}_0 = \frac{\vec{\tilde{k}} \times \vec{E}_0}{\omega}, \tag{1.17c}$$

$$\vec{E}_0 = -\frac{\vec{\tilde{k}} \times \vec{B}_0}{\mu\epsilon\omega}. \tag{1.17d}$$

Equations (1.17a) and (1.17b) specify the transversality condition of the complex field amplitudes $\vec{E}_0$ and $\vec{B}_0$. However, it must be understood that the electric and magnetic fields are transverse to the real wave vector $\vec{k}$ only when the medium is non-absorbing ($\vec{a} = 0$). Combining Eqs (1.17c) and (1.17d) and making use of the vector triple product, we get

$$\tilde{k}^2 = \vec{\tilde{k}} \cdot \vec{\tilde{k}} = \mu\epsilon\omega^2 = \tilde{n}^2 \frac{\omega^2}{c^2}, \tag{1.18a}$$

where

$$\tilde{n}^2 = \mu\epsilon c^2. \tag{1.18b}$$

The real and imaginary parts of the complex refractive index

$$\tilde{n} = n + \mathrm{i}\kappa \tag{1.19}$$

are known as the refractive and extinction indices of the medium, respectively. The real and imaginary parts of the complex wave vector $\vec{\tilde{k}}$ and complex refractive index $\tilde{n}$ satisfy the following relations:

$$k^2 - a^2 = (n^2 - \kappa^2)\frac{\omega^2}{c^2}, \tag{1.20a}$$

$$\vec{k} \cdot \vec{a} = n\kappa\frac{\omega^2}{c^2} \cdot \tag{1.20b}$$

It should be noted that in place of permittivity and permeability, the complex refractive index now describes the bulk properties of an optical material.

1.3.2 Spherical and Cylindrical Wave Solutions

A point source embedded in an isotropic medium generates a spherical wave which propagates radially outward. The surfaces of constant phases for a spherical wave are spherical, centered at the source point. The scalar electric field of a harmonic spherical wave in the complex notation has the form

$$E(r) = \frac{A}{r}\mathrm{e}^{\mathrm{i}(kr-\omega t)}, \tag{1.21a}$$

where A is the amplitude of the spherical wave at unit distance from the point source. The $1/r$ dependence of the field can be easily derived by integrating the wave equation after expressing it in spherical polar coordinates. However, this dependence follows from consideration of energy conservation. Equation (1.21a) represents a diverging or an expanding spherical wave diverging from point $r = 0$, and the spherical wave converging to point $r = 0$ is

$$E(r) = \frac{A}{r}\mathrm{e}^{\mathrm{i}(-kr-\omega t)}. \tag{1.21b}$$

The harmonic cylindrical wave solutions of the wave equation have the form

$$E(r) = \frac{A}{\sqrt{r}}\mathrm{e}^{\mathrm{i}(\pm kr-\omega t)}, \tag{1.21c}$$

where the wavefronts are in the form of coaxial cylindrical surfaces travelling outward from an infinite line source at $r = 0$ or travelling inward to converge on a line at $r = 0$.

1.3.3 Beam-Like Solutions

Laser light possesses a high degree of directionality resembling closely the directionality of a plane wave. But unlike for a plane wave, the field amplitude of laser light decreases rapidly in the transverse plane. Laser light diverges as it propagates, but for short distances the divergence of laser light is much smaller than the divergence of a spherical wave. Of course, laser light is not monochromatic but it is the closest approximation we have for monochromatic light. We now seek a monochromatic solution of the wave equation which is highly directional and possesses a low degree of divergence. It is hoped that such a solution may provide at least an approximate description of laser light. Here, we disregard the fact that the wave equation (1.11) is a vector equation. Instead, we treat the electric and magnetic fields as scalar fields. By doing so, we lose all information about the state of polarization of light to which this solution may correspond. The solution may still be useful to describe interference and diffraction phenomena. We begin by requiring that the beam-like solution be monochromatic, so that

$$E(\vec{r}, t) = E(\vec{r})\mathrm{e}^{-\mathrm{i}\omega t}.$$

On substituting this solution, the wave equation (1.11a) reduces to Helmholtz equation

$$(\nabla^2 + k^2)E(\vec{r}) = 0, \tag{1.22}$$

where

$$k^2 = \mu\epsilon\omega^2 = \omega^2/v^2 = n^2\frac{\omega^2}{c^2}.$$

The propagation vector and index of refraction are assumed real in the present context. To retain the beam-like character of the solution, we write

$$E(\vec{r}) = \varepsilon(\vec{r})\mathrm{e}^{\mathrm{i}kz}. \tag{1.23}$$

The wave propagates in the z-direction with wave number $k = n(\omega/c)$. Noting that

$$\frac{\partial^2}{\partial z^2}\left(\varepsilon(\vec{r})\mathrm{e}^{\mathrm{i}kz}\right) = \left[\frac{\partial^2}{\partial z^2} + 2\mathrm{i}k\frac{\partial}{\partial z} - k^2\right]\varepsilon(\vec{r})\mathrm{e}^{\mathrm{i}kz},$$

Eq. (1.22) can be recast into the form

$$\nabla_t^2\varepsilon(\vec{r}) + \frac{\partial^2\varepsilon(\vec{r})}{\partial z^2} + 2\mathrm{i}k\frac{\partial\varepsilon(\vec{r})}{\partial z} = 0, \tag{1.24}$$

where

$$\nabla_t^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

Making use of the slowly varying envelope approximation (SVEA)

$$\frac{\partial^2 \varepsilon(\vec{r})}{\partial z^2} \ll k \frac{\partial \varepsilon(\vec{r})}{\partial z},$$

Eq. (1.24) can be approximated to

$$\nabla_t^2 \varepsilon(\vec{r}) + 2\mathrm{i}k \frac{\partial \varepsilon(\vec{r})}{\partial z} = 0. \tag{1.25}$$

The SVEA ensures slow variation (on the wavelength scale) of the field amplitude $\varepsilon(r)$ and its derivatives in the direction of propagation. However, appreciable changes in the amplitude over long distances are still permitted. Equation (1.25) admits many beam-like solutions. We look for the one which manifests cylindrical symmetry about the direction of propagation. This may be the simplest, but not the only interesting beam-like solution the wave equation possesses. For the present, it suffices to solve the equation

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial \varepsilon(\vec{r})}{\partial \rho}\right) + 2\mathrm{i}k \frac{\partial \varepsilon(\vec{r})}{\partial z} = 0, \tag{1.26}$$

where $\rho = (x^2 + y^2)^{1/2}$. A possible solution to this equation may have the form

$$\varepsilon(\rho, z) = A\, \mathrm{e}^{\mathrm{i}[p(z) + \frac{1}{2}(k\rho^2)/(q(z))]}, \tag{1.27}$$

where A is a constant. For real $p(z)$ and $q(z)$, the beam intensity is independent of ρ and z. This is not the kind of solution we are seeking. Hence, we expect either one or both of these functions to be complex. Substituting Eq. (1.27) into Eq. (1.26) gives

$$2k\left(\frac{\mathrm{i}}{q(z)} - \frac{\mathrm{d}p(z)}{\mathrm{d}z}\right) + \frac{k^2\rho^2}{q^2(z)}\left(\frac{\mathrm{d}q(z)}{\mathrm{d}z} - 1\right) = 0. \tag{1.28}$$

This equation is satisfied if

$$\frac{\mathrm{d}q(z)}{\mathrm{d}z} = 1, \tag{1.29a}$$

$$\frac{\mathrm{d}p(z)}{\mathrm{d}z} = \frac{\mathrm{i}}{q(z)}. \tag{1.29b}$$

The solution of Eq. (1.29a) is

$$q(z) = z - \mathrm{i}z_0. \tag{1.30}$$

For convenience, the constant of integration has been taken as $-\mathrm{i}z_0$. Integration of Eq. (1.29b) yields

$$p(z) = \mathrm{i}\ \ln(1 + \mathrm{i}z/z_0), \tag{1.31}$$

where the constant of integration has been chosen to make $p(0) = 0$. With this choice, this beam-like solution has exactly the phase (but not the amplitude) of the plane wave at $z = 0$. In other words, the wavefront at $z = 0$ is planar. Equation (1.31) can be expressed as

$$\begin{aligned} \mathrm{e}^{\mathrm{i}pz} &= \left(1 + \mathrm{i}\frac{z}{z_0}\right)^{-1} \\ &= \frac{1}{\sqrt{1 + \frac{z^2}{z_0^2}}}\, \mathrm{e}^{-\mathrm{i}\phi(z)}, \end{aligned} \tag{1.32}$$

where $\phi(z) = \tan^{-1} z/z_0$. Equation (1.30) can be written in an equivalent form

$$\begin{aligned} \frac{1}{q(z)} &= \frac{z}{z^2 + z_0^2} + \mathrm{i}\frac{z_0}{z^2 + z_0^2} \\ &= \frac{1}{R(z)} + \frac{2\mathrm{i}}{k}\frac{1}{w^2(z)}, \end{aligned} \tag{1.33}$$

where

$$R(z) = z + \frac{z_0^2}{z}, \tag{1.34a}$$

$$w^2(z) = w_0^2\left(1 + z^2/z_0^2\right), \tag{1.34b}$$

$$w_0^2 = \frac{2z_0}{k}. \tag{1.34c}$$

Combining these results, the beam-like solution of the wave equation possessing cylindrical symmetry about the direction of propagation can be written as

$$E(\vec{r}, t) = A\frac{w_0}{w(z)}\, \mathrm{e}^{-\rho^2/w^2(z)}\, \mathrm{e}^{\mathrm{i}k\rho^2/2R(z)}\, \mathrm{e}^{\mathrm{i}(kz - \phi(z) - \omega t)}, \tag{1.35a}$$

$$= A\frac{w_0}{w(z)}\, \mathrm{e}^{-\rho^2/w^2(z)}\, \mathrm{e}^{\mathrm{i}k(z + (\rho^2/2R(z)))}\, \mathrm{e}^{-\mathrm{i}\phi(z)}\mathrm{e}^{-\mathrm{i}\omega t}. \tag{1.35b}$$

The two equivalent expressions (1.35a) and (1.35b) have been written to bring out two complementary features of the beam-like solution. The phase factor $(kz - \phi(z) - \omega t)$ in Eq. (1.35a) reminds us of the plane wave solution since $\phi(z)$ is a slowly varying function of z, changing from zero to $\pi/4$ as z goes from zero to z_0. On the other hand, for visible light, kz varies by nearly 10^5 radians over a distance of 1 cm. However, the solution differs from a plane wave because the amplitude of the wave does not remain constant. The expression (1.35b), on the other hand, possesses some implicit resemblance to a spherical wave. The phase factor $k(z+\rho^2/2R(z))$ will be shown to approximate the phase factor kr of a spherical wave in the limit of large r. Furthermore, $w(z)$ varies linearly with z for large z suggesting an inverse dependence of the amplitude on distance as for a spherical wave. But for z, not too large, this solution has much lower divergence as compared to the divergence of a spherical wave. The amplitude

$$E_0(\vec{r}) = A\frac{w_0}{w(z)}\,\mathrm{e}^{-(x^2+y^2)/w^2(z)}$$

of the beam-like solution varies with x, y, z. For a fixed value of z, it has a Gaussian profile in the transverse plane. The amplitude falls to $1/e$ of its maximum value at a distance $\rho = (x^2+y^2)^{1/2} = w(z)$ from the axis of symmetry (Fig. 1.2).

The transverse profile of the beam-like solution changes as the wave propagates. It has minimum spread at $z = 0$. The width of the transverse profile of the beam increases non-linearly with z on either side of the point $z = 0$. However,

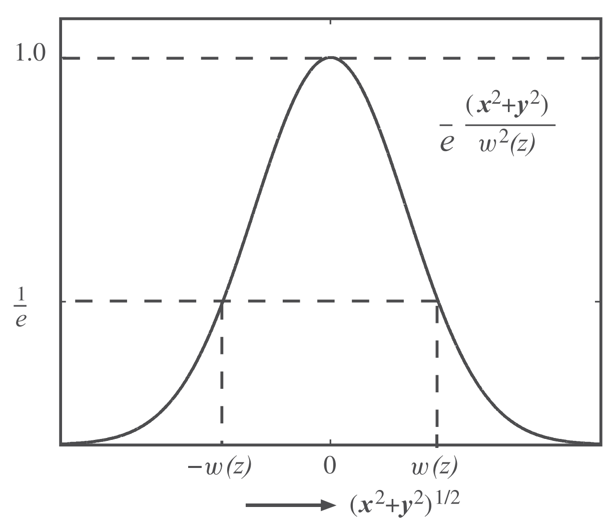


Fig. 1.2: Gaussian profile of the amplitude of the beam-like solution.

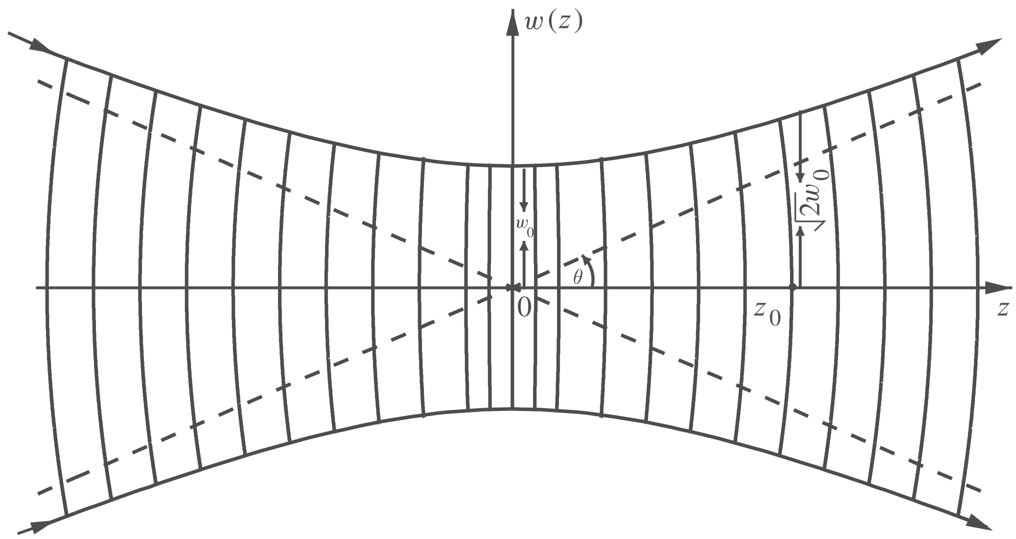


Fig. 1.3: Variation of the transverse profile of the beam-like solution; w_0 is beam waist and z_0 is Rayleigh range.

for $|z| \gg z_0$, the transverse profile shows a linear dependence on z. This behavior of the solution is shown in Fig. 1.3.

We next consider the spatial phase of the wave,

$$\Phi(x, y, z) = k\left(z + \frac{x^2 + y^2}{2R(z)}\right). \tag{1.36a}$$

This phase is obviously not constant for a given value of z. The equiphase surfaces are curved, but not necessarily spherical (Fig. 1.3). For comparison, we write the spatial phase of a spherical wave in the limit $x, y \ll z$:

$$\begin{aligned}\Phi_{\text{sph}}(x, y, z) &= kr \\ &= k[x^2 + y^2 + z^2]^{1/2} \\ &\approx k[z + \frac{x^2 + y^2}{2z}].\end{aligned} \tag{1.36b}$$

Only the first term in the binomial expansion has been retained. The expressions (1.36a) and (1.36b) are similar since $R(z) \sim z$ for large z. One may therefore conclude that for points in the transverse plane, not too far from the axis of symmetry, the curvature of the equiphase surface of the beam-like solution approaches sphericity for large values of z. It is tempting to identify the factor $1/R(z)$ with the curvature of the equiphase surface. The curvature changes continuously from planar at $z = 0$ to near-spherical for large z, taking more complex forms in the intermediate region. Sections of these surfaces are shown in Fig. 1.3. The curvature changes sign as the point $z = 0$ is crossed. The intensity distribution

$$I(x, y, z) = \left(\frac{1}{2}\epsilon_0 c\right) A^2 \left(\frac{w_0}{w(z)}\right)^2 \mathrm{e}^{-2(x^2+y^2)/(w^2(z))} \tag{1.37}$$

of the beam-like solution has Gaussian profile in the transverse plane with $1/e^2$ half-width which varies from w_0 at $z = 0$ to $w = \sqrt{2}w_0$ at $z = z_0$ and increases approximately linearly for large values of $|z|$. The beam in any transverse plane will have the appearance of a bright round spot with *spot size* ($1/e^2$ beam radius) $w(z)$. At the beam waist ($z = 0$), the spot size has the least value (w_0). The distance z_0 over which the spot size changes from w_0 to $\sqrt{2}w_0$ is known as the Rayleigh range. The beam divergence, defined asymptotically, is

$$\theta(\text{divergence}) = \lim_{z\to\infty} \frac{\mathrm{d}w(z)}{\mathrm{d}z} = \frac{w_0}{z_0} = \frac{\lambda_v}{\pi n w_0},$$

where λ_v is wavelength of light in a vacuum and n is refractive index of the medium. Typical divergence angle of the beam of a commercial laser is in milliradians.

As mentioned earlier, we have considered only the lowest order beam-like solution (TEM_{00}mode) of the wave equation which has been found to resemble in some way a plane wave for $z \to 0$ and a spherical wave for $z \to \pm\infty$. Higher order solutions of the wave equation with beam-like character also exist. They are described in terms of the Hermite polynomials [1.1, 1.2].

1.4 HOMOGENEOUS AND INHOMOGENEOUS WAVES

A vacuum is a perfectly transparent medium for the entire range of the electromagnetic spectrum. Other media may approach complete transparency over limited spectral bandwidths. Perfect transparency exists in an optical medium when the index of refraction is purely real ($\kappa = 0$). This need not necessarily imply a purely real propagation vector (a non-absorbing medium). For perfect transparency, Eq. (1.20b) requires

$$\vec{k} \cdot \vec{a} = 0. \tag{1.38}$$

This condition can be met in two ways. The attenuation vector may be a null vector ($\vec{a} = 0$), in which case, the plane wave solution takes the form

$$\begin{aligned} \vec{E} &= \vec{E}_0 \, \mathrm{e}^{\mathrm{i}(\vec{k} \cdot \vec{r} - \omega t)}, \\ \vec{B} &= \vec{B}_0 \, \mathrm{e}^{\mathrm{i}(\vec{k} \cdot \vec{r} - \omega t)}, \end{aligned} \tag{1.39}$$

where $\vec{k}$ is now a real vector of magnitude

$$k = n\frac{\omega}{c}. \tag{1.40}$$

These fields represent a homogeneous plane wave with coincident surfaces of constant amplitude ($\vec{E}_0 =$ constant, $\vec{B}_0 =$ constant) and constant phase ($\vec{k} \cdot \vec{r} =$ constant). These surfaces are planes perpendicular to the real wave vector $\vec{k}$. Equations (1.39) represent a wave with unchanging amplitude propagating with speed

$$v = \frac{\omega}{k} = \frac{c}{n}. \tag{1.41}$$

In this case, Eqs. (1.17) have clear physical interpretation. The real and imaginary parts of the $\vec{E}$ and $\vec{B}$ fields are transverse to the direction of propagation. It should be understood that we have used the complex notation for the fields only for the sake of convenience. The physical electric and magnetic fields being real are not only transverse to the direction of propagation, but are also transverse to each other in the present case. Such a wave is called a TEM wave, where TEM stands for transverse electric and magnetic fields (Fig. 1.4). The electric and magnetic fields remain in phase and their amplitudes are related by

$$B_0 = \frac{n}{c} E_0. \tag{1.42}$$

For a perfectly transparent medium ($\kappa = 0$), the condition (1.38) can also be met for a non-zero value of the attenuation vector $\vec{a}$ provided the real and imaginary parts of the complex wave vector $\vec{\tilde{k}}$ are orthogonal to each other. In this case, the plane wave solution takes the form

$$\vec{E}(\vec{r}, t) = \vec{E}_0 \, \mathrm{e}^{-\vec{a} \cdot \vec{r}} \, \mathrm{e}^{\mathrm{i}(\vec{k} \cdot \vec{r} - \omega t)}. \tag{1.43}$$

The wave now propagates in the direction of $\vec{k}$ with somewhat diminished velocity as compared to the velocity of the homogeneous wave ($\vec{a} = 0$). The surfaces of constant phase and constant amplitude are no longer coincident. The surfaces

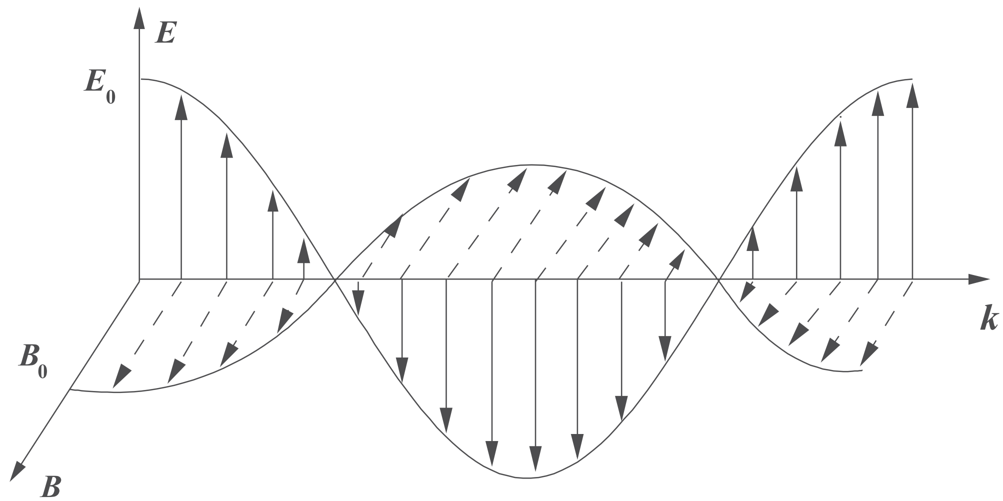
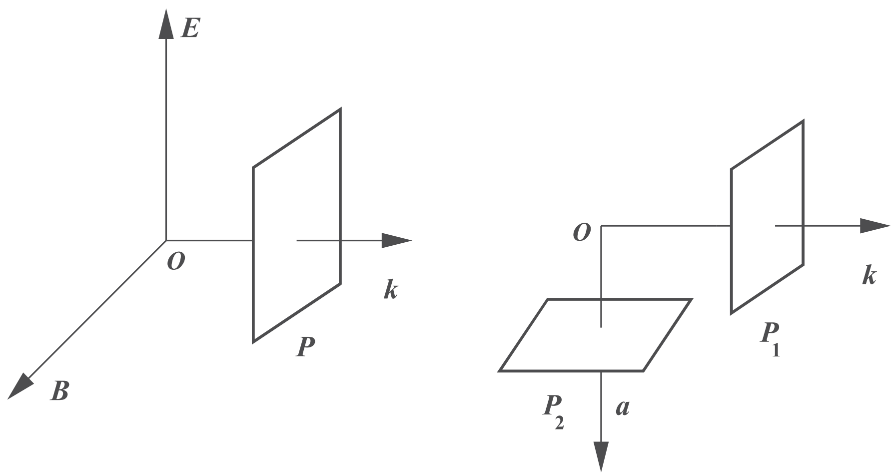


Fig. 1.4: A homogeneous harmonic plane wave; electric and magnetic fields are transverse to the direction of propagation and also to each other.

of constant phase remain perpendicular to the direction of propagation $\vec{k}$, but the surfaces of constant amplitude ($\vec{E}_0\ \mathrm{e}^{-\vec{a}\cdot\vec{r}} =$ constant) are now planes perpendicular to the direction of the attenuation vector $\vec{a}$ since $\vec{a} \cdot \vec{r}$ remains constant in a plane normal to $\vec{a}$. The amplitude of the wave decreases in the direction of $\vec{a}$. This is the inhomogeneous wave. Figure 1.5 compares a homogeneous wave with an inhomogeneous wave of this kind. A wave is inhomogeneous if the surfaces of constant amplitude and constant phase are not coincident. The field configurations are not easy to visualize for the inhomogeneous waves. For the TE mode, the real and imaginary parts of the electric field $\vec{E}$ are perpendicular to the plane containing the propagation vector $\vec{k}$ and attenuation vector $\vec{a}$. It can be shown (see Problem 1.4) that the magnetic field for the TE mode is elliptically polarized. For the TM mode, the real and imaginary parts of the magnetic field $\vec{B}$ are perpendicular to the plane of $\vec{k}$ and $\vec{a}$. Any field configuration can be expressed as a superposition of TE and TM modes. An example of an inhomogeneous wave is the evanescent wave to be considered later in this chapter.

For the more general case of non-zero extinction index κ, the attenuation vector $\vec{a}$ is not normal to the propagation vector $\vec{k}$ and the amplitude of the inhomogeneous wave decreases in the direction of propagation as well. The surfaces of constant phase and constant amplitude are neither coincident nor orthogonal. Electromagnetic waves in metals behave in this manner.



(a) (b)

Fig. 1.5: (a) A homogeneous plane wave; planes of constant phase and planes of constant amplitude are coincident (P). (b) An inhomogeneous plane wave; planes of constant phase (P_1) are perpendicular to propagation vector $\vec{k}$ and planes of constant amplitude (P_2) are perpendicular to attenuation vector $\vec{a}$.

1.5 ENERGY DENSITY AND POYNTING VECTOR

A wave carries energy as it propagates in a medium. The instantaneous energy density stored in the medium due to the presence of the wave is given by[1]

$$u = \frac{1}{2}\epsilon(E^{(\mathrm{r})})^2 + \frac{1}{2\mu}(B^{(\mathrm{r})})^2 \tag{1.44a}$$

and the instantaneous energy crossing per unit area per unit time is given by the Poynting vector

$$\vec{S} = \frac{\vec{E}^{(\mathrm{r})} \times \vec{B}^{(\mathrm{r})}}{\mu}, \tag{1.44b}$$

where $\vec{E}^{(\mathrm{r})}$ and $\vec{B}^{(\mathrm{r})}$ are real time-dependent fields. The more relevant quantities for light fields are their time averaged values. In the complex notation,

$$\langle u \rangle = \frac{1}{4} Re\left[\epsilon\, \vec{E} \cdot \vec{E}^* + \frac{1}{\mu}\vec{B} \cdot \vec{B}^*\right]$$

[1] Introduction to Electrodynamics by David J. Griffiths.

and

$$\langle \vec{S} \rangle = \frac{1}{2\mu} Re[\vec{E} \times \vec{B}^*].$$

For a propagating TEM wave,

$$\epsilon \vec{E} \cdot \vec{E}^* = \frac{1}{\mu} \vec{B} \cdot \vec{B}^*,$$

so that

$$\langle u \rangle = \frac{1}{2} \epsilon \vec{E} \cdot \vec{E}^* = \epsilon \langle (E^{(r)})^2 \rangle$$

and

$$\langle \vec{S} \rangle = \frac{1}{2\mu} Re \langle \vec{E} \times \vec{B}^* \rangle = \frac{1}{2\mu v} EE^* \hat{s},$$

where the symbol $\langle \; \rangle$ represents the average over a time needed to make a measurement which is much longer than the period of a light wave and $\hat{s}$ is a unit vector in the direction of $\vec{S}$. The intensity of a wave, defined as the magnitude of the time averaged Poynting vector, is given by

$$I = \langle S \rangle = \frac{1}{2\mu v} EE^* = \frac{1}{2} \epsilon v EE^*, \tag{1.45}$$

where v is the velocity of the wave in the medium. The expression

$$I = \frac{1}{2} n \epsilon_0 c |E|^2, \tag{1.46}$$

commonly used in literature makes the reasonable assumption of $\mu \approx \mu_0$ for an optically transparent medium of refractive index n. A useful relation between the energy density and intensity of a plane wave is

$$I = v \langle u \rangle. \tag{1.47}$$

1.6 BOUNDARY CONDITIONS

We have so far been considering wave propagation in a source-free infinite homogeneous medium. In practice, one encounters wave propagation in a medium of finite extent. We need to address ourselves to the question of matching the solutions of the wave equation at the interface between two media. It is convenient to assume a plane boundary separating the two media. This assumption may actually be not as restrictive as it appears at first sight. As mentioned earlier, the macroscopically averaged electric and magnetic fields satisfy Gauss and Stokes theorems everywhere in the two media including the region surrounding the boundary between them. The restrictions imposed by these theorems on the fields on the two sides of the interface are called the boundary conditions.

1.6.1 Continuity of the Normal Components

Consider a small pillbox around the interface between two media of permittivities ϵ_1 and ϵ_2 (Fig. 1.6a). The height h of the pillbox is infinitesimally small bringing the flat surfaces of the pillbox very close, but on the opposite sides of the boundary. We apply Gauss' theorem

$$\oiint_S \vec{D} \cdot \mathrm{d}\vec{A} = \iiint_V \nabla \cdot \vec{D} \, \mathrm{d}V$$

to the displacement field $\vec{D}$ over this pillbox. The integral on the left-hand side is over the closed surface S bounding the volume V. The volume integral on the right-hand side vanishes when the volume of the pillbox approaches zero as $h \to 0$. In the same limit, the contribution to the surface integral from the curved surface of the pillbox is vanishingly small. The flat surfaces of the pillbox are taken sufficiently small so that the normal component of the displacement field contributing to the surface integral in each medium remains constant. Therefore,

$$\epsilon_1 \vec{E}_1 \cdot \hat{n}' + \epsilon_2 \vec{E}_2 \cdot \hat{n} = 0,$$

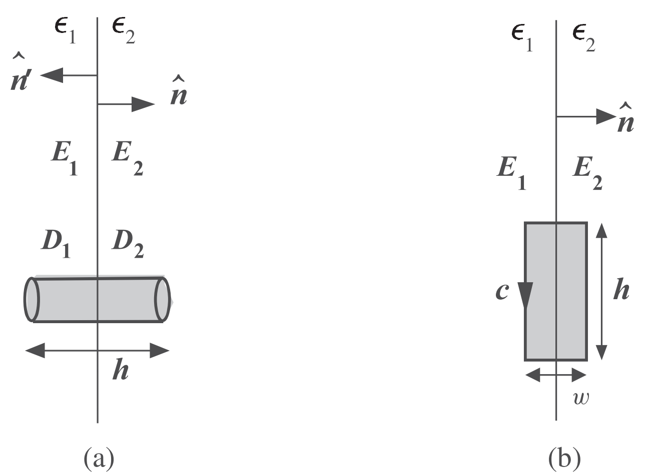


Fig. 1.6: Plane boundary between two homogeneous media.

where the unit vectors $\hat{n}'$ and $\hat{n}$ are normal to the boundary as shown in the figure. With $\hat{n}' = -\hat{n}$, the above condition, expressed as

$$\epsilon_1 \vec{E}_1 \cdot \hat{n} = \epsilon_2 \vec{E}_2 \cdot \hat{n}, \tag{1.48a}$$

is a statement of the continuity of the normal components of the displacement fields across the boundary between two homogeneous media. A similar condition holds for the normal components of the $\vec{B}$ fields, i.e.,

$$\vec{B}_1 \cdot \hat{n} = \vec{B}_2 \cdot \hat{n}. \tag{1.48b}$$

1.6.2 Continuity of the Tangential Components

Next, we apply Stokes' theorem

$$\oint_c \vec{E} \cdot \mathrm{d}\vec{l} = \int\int_{\Sigma} \nabla \times \vec{E} \cdot \mathrm{d}\vec{A}$$
$$= -\frac{\partial}{\partial t} \int\int_{\Sigma} \vec{B} \cdot \mathrm{d}\vec{A}$$

to the electric field, where the closed path c encloses the boundary between the two media as shown in Fig. 1.6b. Here, $\sum$ is a surface bounded by the closed path c. The side h of the rectangular path is taken sufficiently small so that the tangential fields do not change appreciably in each medium over the paths parallel to the boundary. The surface integral on the right-hand side vanishes as the width w of the rectangular path approaches zero, leading to the continuity of the tangential components of the electric fields across the boundary, i.e.,

$$\vec{E}_1 \times \hat{n} = \vec{E}_2 \times \hat{n}. \tag{1.48c}$$

The continuity of the tangential components of the $\vec{H}$ fields can be shown in a similar manner. So that,

$$\vec{H}_1 \times \hat{n} = \vec{H}_2 \times \hat{n}$$

or equivalently

$$\frac{\vec{B}_1}{\mu_1} \times \hat{n} = \frac{\vec{B}_2}{\mu_2} \times \hat{n}, \tag{1.48d}$$

where μ_1 and μ_2 are the permeabilities of the two media. We may make the reasonable assumption that for the optically transparent media $\mu_1 \approx \mu_2 = \mu_0$. These four relations (Eqs 1.48) constitute the boundary conditions which must be satisfied across an interface between two homogeneous media.

1.7 REFLECTION AND TRANSMISSION AT A BOUNDARY

The boundary conditions obtained in Section 1.6 can be used to obtain relationships among the amplitudes of the reflected, transmitted and incident waves at the boundary between two homogeneous media (Fig. 1.7). This exercise can be quite tedious. Our approach here is to avoid mathematical complications as far as possible, but at the same time not to miss the essential features of what goes on at the interface. Following Stone [1.3], we consider light incidence from a perfectly transparent ($\kappa_1 = 0$) and non-absorbing ($\vec{a}_1 = 0$) medium of refractive index n_1 to a medium for which the refractive index $\tilde{n}$ and wave vector $\tilde{\vec{k}}$ may be complex.

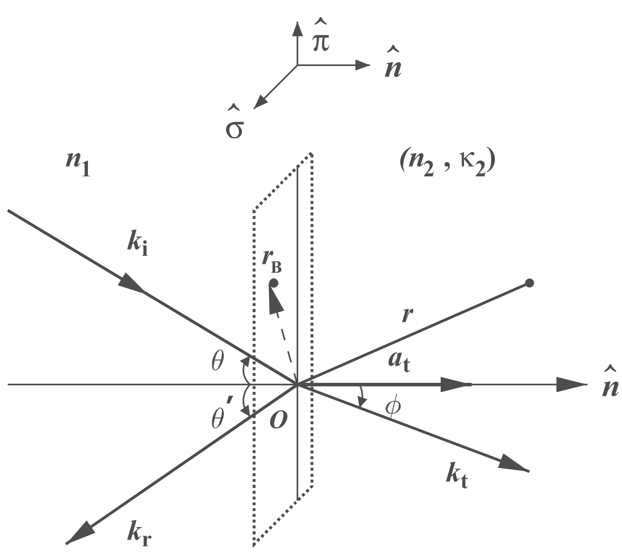


Fig. 1.7: Reflection and transmission of a wave at a plane boundary.

The incident wave is therefore homogeneous. We can anticipate the reflected wave to be homogeneous as well, but the transmitted wave in general will be inhomogeneous. Accordingly, the fields in the two media can be expressed as Incident wave:

$$\begin{aligned} \vec{E}_{\mathrm{in}} &= \vec{E}_{\mathrm{i}}\ \mathrm{e}^{\mathrm{i}(\vec{k}_{\mathrm{i}}\cdot\vec{r}-\omega t)}, \\ \vec{B}_{\mathrm{in}} &= \vec{B}_{\mathrm{i}}\ \mathrm{e}^{\mathrm{i}(\vec{k}_{\mathrm{i}}\cdot\vec{r}-\omega t)}, \end{aligned} \tag{1.49a}$$

Reflected wave:

$$\begin{aligned} \vec{E}_{\mathrm{re}} &= \vec{E}_{\mathrm{r}}\ \mathrm{e}^{\mathrm{i}(\vec{k}_{\mathrm{r}}\cdot\vec{r}-\omega' t)}, \\ \vec{B}_{\mathrm{re}} &= \vec{B}_{\mathrm{r}}\ \mathrm{e}^{\mathrm{i}(\vec{k}_{\mathrm{r}}\cdot\vec{r}-\omega' t)}, \end{aligned} \tag{1.49b}$$

Transmitted wave:

$$\begin{aligned} \vec{E}_{\mathrm{tr}} &= \vec{E}_{\mathrm{t}}\ \mathrm{e}^{\mathrm{i}[(\vec{k}_{\mathrm{t}}+\mathrm{i}\vec{a}_{\mathrm{t}})\cdot\vec{r}-\omega'' t]}, \\ \vec{B}_{\mathrm{tr}} &= \vec{B}_{\mathrm{t}}\ \mathrm{e}^{\mathrm{i}[(\vec{k}_{\mathrm{t}}+\mathrm{i}\vec{a}_{\mathrm{t}})\cdot\vec{r}-\omega'' t)]}, \end{aligned} \tag{1.49c}$$

where the amplitude vectors $\vec{E}_{\mathrm{i}}$, $\vec{B}_{\mathrm{i}}$, $\vec{E}_{\mathrm{r}}$, $\vec{B}_{\mathrm{r}}$, $\vec{E}_{\mathrm{t}}$, and $\vec{B}_{\mathrm{t}}$ are in general complex. The boundary conditions (1.48c) and (1.48d) require

$$\left[\vec{E}_{\mathrm{i}}\ \mathrm{e}^{\mathrm{i}(\vec{k}_{\mathrm{i}}\cdot\vec{r}_{\mathrm{B}}-\omega t)}+\vec{E}_{\mathrm{r}}\ \mathrm{e}^{\mathrm{i}(\vec{k}_{\mathrm{r}}\cdot\vec{r}_{\mathrm{B}}-\omega' t)}\right]\times\hat{n}=\left[\vec{E}_{\mathrm{t}}\ \mathrm{e}^{\mathrm{i}[(\vec{k}_{\mathrm{t}}+\mathrm{i}\vec{a}_{\mathrm{t}})\cdot\vec{r}_{\mathrm{B}}-\omega'' t]}\right]\times\hat{n} \tag{1.50a}$$

and

$$\left[\vec{B}_{\mathrm{i}}\ \mathrm{e}^{\mathrm{i}(\vec{k}_{\mathrm{i}}\cdot\vec{r}_{\mathrm{B}}-\omega t)}+\vec{B}_{\mathrm{r}}\ \mathrm{e}^{\mathrm{i}(\vec{k}_{\mathrm{r}}\cdot\vec{r}_{\mathrm{B}}-\omega' t)}\right]\times\hat{n}=\left[\vec{B}_{\mathrm{t}}\ \mathrm{e}^{\mathrm{i}[(\vec{k}_{\mathrm{t}}+\mathrm{i}\vec{a}_{\mathrm{t}})\cdot\vec{r}_{\mathrm{B}}-\omega'' t]}\right]\times\hat{n}. \tag{1.50b}$$

Here, $\vec{r}_{\mathrm{B}}$ is the position vector of a point in the plane of the boundary with respect to a suitably chosen origin also lying in this plane. These conditions must be satisfied at all times and for all points lying on the infinite boundary plane. This can be ensured if all phase factors associated with the fields are equal. Hence

$$\omega''=\omega'=\omega, \tag{1.51a}$$

$$\vec{a}_{\mathrm{t}}\cdot\vec{r}_{\mathrm{B}}=0 \tag{1.51b}$$

and

$$\vec{k}_{\mathrm{i}}\cdot\vec{r}_{\mathrm{B}}=\vec{k}_{\mathrm{r}}\cdot\vec{r}_{\mathrm{B}}=\vec{k}_{\mathrm{t}}\cdot\vec{r}_{\mathrm{B}}\,. \tag{1.51c}$$

The boundary conditions therefore require the incident, reflected, and transmitted waves to have the same frequency. The magnitudes of the wave vectors of the incident and reflected waves, being in the same medium, are equal, i.e.,

$$|\vec{k}_i| = |\vec{k}_r| = k = n_1 \frac{\omega}{c}. \tag{1.51d}$$

Equation (1.51b) requires the attenuation vector in the second medium to be directed along the normal to the plane of the boundary, i.e.,

$$\vec{a}_t = a_t \hat{n}. \tag{1.51e}$$

The condition (1.51c) can be re-expressed as,

$$\vec{k}_i \cdot \hat{n} \times \vec{r} = \vec{k}_r \cdot \hat{n} \times \vec{r} = \vec{k}_t \cdot \hat{n} \times \vec{r}, \tag{1.51f}$$

where $\hat{n} \times \vec{r}$ is a convenient representation for vector $\vec{r}_B$ lying in the plane of the boundary in terms of an arbitrary position vector $\vec{r}$ (see Fig. 1.7). Manipulation of the scalar triple product leads to the important result:

$$\vec{k}_i \times \hat{n} = \vec{k}_r \times \hat{n} = \vec{k}_t \times \hat{n}. \tag{1.52}$$

This is the statement of the coplanarity of the wave vectors $\vec{k}_i$, $\vec{k}_r$, $\vec{k}_t$, and the normal $\hat{n}$ to the plane of the interface. In addition, Eq. (1.52) requires

$$\theta' = \theta \tag{1.53a}$$

and

$$k_t \sin \phi = k \sin \theta, \tag{1.53b}$$

where θ, θ', and ϕ are the angles of incidence, reflection, and refraction, respectively. These equations ensure the equality of the angles of incidence and reflection, but leave the angle of refraction ϕ and magnitude k_t of the real part of the propagation vector in the second medium undetermined – only the product $k_t \sin \phi$ is determined. Equations (1.52) and (1.53b) describe the laws of reflection and refraction of light across an interface. Combining Eqs (1.16), (1.18), and (1.19), we get

$$(k_t \cos \phi + \mathrm{i} a_t)^2 + (k_t \sin \phi)^2 = \frac{\omega^2}{c^2} (n_2 + \mathrm{i}\kappa_2)^2. \tag{1.54}$$

Knowing n_1, n_2, and κ_2, Eqs (1.53b) and (1.54) suffice to determine ϕ, k_t, and a_t. With the equality of the phase factors guaranteed by Eqs (1.51), the restrictions (Eqs 1.50a and 1.50b) on the fields go over to the restrictions on the corresponding field amplitudes. Therefore,

$$(\vec{E}_\text{i} + \vec{E}_\text{r})_\text{boundary} \times \hat{n} = (\vec{E}_\text{t})_\text{boundary} \times \hat{n}, \tag{1.55a}$$

$$(\vec{B}_\text{i} + \vec{B}_\text{r})_\text{boundary} \times \hat{n} = (\vec{B}_\text{t})_\text{boundary} \times \hat{n}. \tag{1.55b}$$

Expressing the electric fields in terms of the Cartesian components, we have

$$\vec{E}_\text{i} = E_{\text{i}n}\hat{n} + E_{\text{i}\pi}\hat{\pi} + E_{\text{i}\sigma}\hat{\sigma}, \tag{1.56a}$$

$$\vec{E}_\text{r} = E_{\text{r}n}\hat{n} + E_{\text{r}\pi}\hat{\pi} + E_{\text{r}\sigma}\hat{\sigma}, \tag{1.56b}$$

$$\vec{E}_\text{t} = E_{\text{t}n}\hat{n} + E_{\text{t}\pi}\hat{\pi} + E_{\text{t}\sigma}\hat{\sigma}, \tag{1.56c}$$

where the unit vectors $\hat{\pi}$, $\hat{\sigma}$, $\hat{n}$ constitute a right-handed Cartesian coordinate system with the unit vectors $\hat{\pi}$ and $\hat{\sigma}$ lying in the plane of the boundary and unit vector $\hat{n}$ pointing normal to it (Fig. 1.7). We can choose the unit vector $\hat{\pi}$ to lie in the plane of incidence (plane containing $\vec{k}_\text{i}$, $\vec{k}_\text{r}$, $\vec{k}_\text{t}$, $\hat{n}$). Similarly, decomposing the propagation vectors of the three waves in the chosen system of coordinates, we have

$$\vec{k}_\text{i} = (k\cos\theta)\hat{n} - (k\sin\theta)\hat{\pi}, \tag{1.57a}$$

$$\vec{k}_\text{r} = -(k\cos\theta)\hat{n} - (k\sin\theta)\hat{\pi}, \tag{1.57b}$$

$$\vec{k}_\text{t} = (k_\text{t}\cos\phi)\hat{n} - (k_\text{t}\sin\phi)\hat{\pi}. \tag{1.57c}$$

The transversality conditions (1.17a,b) require

$$E_{\text{i}n} = E_{\text{i}\pi}\tan\theta, \tag{1.58a}$$

$$E_{\text{r}n} = -E_{\text{r}\pi}\tan\theta, \tag{1.58b}$$

$$E_{\text{t}n} = \frac{k_\text{t}\sin\phi}{k_\text{t}\cos\phi + \text{i}a_\text{t}}E_{\text{t}\pi}. \tag{1.58c}$$

Using Eqs (1.17), (1.56), and (1.57), the magnetic field vectors associated with the incident, reflected, and transmitted waves can be expressed in terms of the components of the corresponding electric field vectors. So that,

$$\vec{B}_{\mathrm{i}} = \frac{k}{\omega}[-(E_{\mathrm{i}\sigma}\sin\theta)\hat{n} - (E_{\mathrm{i}\sigma}\cos\theta)\hat{\pi} + (E_{\mathrm{i}n}\sin\theta + E_{\mathrm{i}\pi}\cos\theta)\hat{\sigma}], \tag{1.59a}$$

$$\vec{B}_{\mathrm{r}} = \frac{k}{\omega}[-(E_{\mathrm{r}\sigma}\sin\theta)\hat{n} + (E_{\mathrm{r}\sigma}\cos\theta)\hat{\pi} + (E_{\mathrm{r}n}\sin\theta - E_{\mathrm{r}\pi}\cos\theta)\hat{\sigma}], \tag{1.59b}$$

$$\begin{aligned}\vec{B}_{\mathrm{t}} = \frac{1}{\omega}[&-(E_{\mathrm{t}\sigma}k_{\mathrm{t}}\sin\phi)\hat{n} - (E_{\mathrm{t}\sigma}k_{\mathrm{t}}\cos\phi + \mathrm{i}E_{\mathrm{t}\sigma}a_{\mathrm{t}})\hat{\pi} \\ &+ (E_{\mathrm{t}\pi}k_{\mathrm{t}}\cos\phi + E_{\mathrm{t}n}k_{\mathrm{t}}\sin\phi + \mathrm{i}E_{\mathrm{t}\pi}a_{\mathrm{t}})\hat{\sigma}].\end{aligned} \tag{1.59c}$$

The field components of the incident wave are determined by its state of polarization and are therefore known. The boundary conditions (1.55a,b) impose the following restrictions on the components of the reflected and transmitted fields:

$$(E_{\mathrm{i}\pi} + E_{\mathrm{r}\pi})_{\mathrm{boundary}} = (E_{\mathrm{t}\pi})_{\mathrm{boundary}}, \tag{1.60a}$$

$$(E_{\mathrm{i}\sigma} + E_{\mathrm{r}\sigma})_{\mathrm{boundary}} = (E_{\mathrm{t}\sigma})_{\mathrm{boundary}}, \tag{1.60b}$$

$$(B_{\mathrm{i}\pi} + B_{\mathrm{r}\pi})_{\mathrm{boundary}} = (B_{\mathrm{t}\pi})_{\mathrm{boundary}}, \tag{1.60c}$$

$$(B_{\mathrm{i}\sigma} + B_{\mathrm{r}\sigma})_{\mathrm{boundary}} = (B_{\mathrm{t}\sigma})_{\mathrm{boundary}}. \tag{1.60d}$$

Equations (1.60c,d) involving the tangential components of the magnetic fields can be expressed in terms of the tangential components of the electric fields:

$$k(E_{\mathrm{i}\sigma} - E_{\mathrm{r}\sigma})\cos\theta = (k_{\mathrm{t}}\cos\phi + \mathrm{i}a_{\mathrm{t}})E_{\mathrm{t}\sigma}, \tag{1.61a}$$

$$\frac{k}{\cos\theta}(E_{\mathrm{i}\pi} - E_{\mathrm{r}\pi}) = \frac{(\vec{k}_{\mathrm{t}} + \mathrm{i}\,\vec{a}_{\mathrm{t}})^2}{k_{\mathrm{t}}\cos\phi + \mathrm{i}a_{\mathrm{t}}}E_{\mathrm{t}\pi}. \tag{1.61b}$$

Equations (1.58), (1.60a,b), and (1.61) can now be solved to obtain the amplitude reflection and transmission coefficients:

$$r_\sigma = \left(\frac{E_{\mathrm{r}\sigma}}{E_{\mathrm{i}\sigma}}\right)_{\mathrm{boundary}} = \frac{k\cos\theta - k_{\mathrm{t}}\cos\phi - \mathrm{i}a_{\mathrm{t}}}{k\cos\theta + k_{\mathrm{t}}\cos\phi + \mathrm{i}a_{\mathrm{t}}}, \tag{1.62a}$$

$$r_\pi = \left(\frac{E_{\mathrm{r}\pi}}{E_{\mathrm{i}\pi}}\right)_{\mathrm{boundary}} = \frac{n_1^2(k_{\mathrm{t}}\cos\phi + \mathrm{i}a_{\mathrm{t}}) - (n_2 + \mathrm{i}\kappa_2)^2 k\cos\theta}{n_1^2(k_{\mathrm{t}}\cos\phi + \mathrm{i}a_{\mathrm{t}}) + (n_2 + \mathrm{i}\kappa_2)^2 k\cos\theta}, \tag{1.62b}$$

$$r_n = \left(\frac{E_{\mathrm{r}n}}{E_{\mathrm{i}n}}\right)_{\mathrm{boundary}} = -r_\pi, \tag{1.62c}$$

$$t_\sigma = \left(\frac{E_{\mathrm{t}\sigma}}{E_{\mathrm{i}\sigma}}\right)_{\text{boundary}} = \frac{2k\cos\theta}{k\cos\theta + k_\mathrm{t}\cos\phi + \mathrm{i}a_\mathrm{t}}, \tag{1.62d}$$

$$t_\pi = \left(\frac{E_{\mathrm{t}\pi}}{E_{\mathrm{i}\pi}}\right)_{\text{boundary}} = \frac{2n_1^2(k_\mathrm{t}\cos\phi + \mathrm{i}a_\mathrm{t})}{n_1^2(k_\mathrm{t}\cos\phi + \mathrm{i}a_\mathrm{t}) + (n_2 + \mathrm{i}\kappa_2)^2 k\cos\theta}, \tag{1.62e}$$

$$t_n = \left(\frac{E_{\mathrm{t}n}}{E_{\mathrm{i}n}}\right)_{\text{boundary}} = \frac{k\cos\theta}{k_\mathrm{t}\cos\phi + \mathrm{i}a_\mathrm{t}} t_\pi. \tag{1.62f}$$

We note that the reflection and transmission coefficients are complex, implying that the reflected and transmitted fields are in general not in phase with the incident field. Some care needs to be exercised to distinguish between the $\hat{n}$- and $\hat{\pi}$-polarizations – both lying in the plane of incidence. Their reflection coefficients have equal magnitudes but are 180° out of phase at all angles of incidence whereas the transmission coefficients for these polarizations differ in phase as well as in magnitude at all angles of incidence.

In the present example, the reflection and transmission coefficients were obtained from the continuity of the tangential components of the fields (Eqs 1.48c,d) at the interface and some intuition concerning the incident and reflected fields in the first medium. In other situations, it may be necessary to use the continuity of the normal components (Eqs 1.48a,b) also.

1.7.1 External Reflections

We first consider the case when light crosses an interface from an optically rare medium to an optically dense medium ($n_1 < n_2$). Reflections under these conditions are known as external reflections. If the second medium is also perfectly transparent ($\kappa_2 = 0$), then Eq. (1.54) when combined with Eq. (1.53b) gives

$$k_\mathrm{t}\cos\phi + \mathrm{i}a_\mathrm{t} = \frac{\omega}{c}(n_2^2 - n_1^2\sin^2\theta)^{1/2}. \tag{1.63}$$

For $n_2 > n_1$, the right-hand side of Eq. (1.63) remains real for all angles of incidence. Therefore, the attenuation vector must vanish, i.e.,

$$a_\mathrm{t} = 0$$

and

$$k_\mathrm{t}\cos\phi = \frac{\omega}{c}(n_2^2 - n_1^2\sin^2\theta)^{1/2}.$$

In this case the transmitted wave in the second medium is also homogeneous with

$$k_\mathrm{t} = n_2\frac{\omega}{c}, \tag{1.64a}$$

and Eq. (1.53b) takes the more familiar form

$$n_2 \sin \phi = n_1 \sin \theta. \tag{1.64b}$$

This is the well-known Snell's law which holds at the interface between two perfectly transparent media under conditions of external reflections ($n_2 > n_1$). It is not obvious at this stage whether Snell's law in its present form will hold when light is incident from an optically more dense medium to an optically less dense medium. Equations (1.53b) and (1.54) may be taken together to represent the more general form of Snell's law. For external reflections, Eqs (1.62) simplify to

$$r_\sigma = \frac{n_1 \cos \theta - n_2 \cos \phi}{n_1 \cos \theta + n_2 \cos \phi}, \tag{1.65a}$$

$$r_\pi = \frac{n_1 \cos \phi - n_2 \cos \theta}{n_1 \cos \phi + n_2 \cos \theta}, \tag{1.65b}$$

$$r_n = -r_\pi, \tag{1.65c}$$

$$t_\sigma = \frac{2n_1 \cos \theta}{n_1 \cos \theta + n_2 \cos \phi}, \tag{1.65d}$$

$$t_\pi = \frac{2n_1 \cos \phi}{n_1 \cos \phi + n_2 \cos \theta}, \tag{1.65e}$$

$$t_n = \frac{n_1 \cos \theta}{n_2 \cos \phi} t_\pi. \tag{1.65f}$$

Equations (1.65) constitute the Fresnel relations. They are applicable when light enters from a perfectly transparent medium of smaller index of refraction into another perfectly transparent medium of higher index of refraction. Some of these relations may differ from the standard form of Fresnel relations given in many texts. We shall return to these differences shortly.

It will be shown in Section 6.5.1 that if the direction of incidence is reversed, i.e., if light enters the medium of index of refraction n_1 from medium of index of refraction n_2, then the new reflection coefficients r'_σ, r'_π and the new transmission coefficients t'_σ, t'_π satisfy the following relationships:

$$r'_\sigma = -r_\sigma, \tag{1.65g}$$

$$r'_\pi = -r_\pi, \tag{1.65h}$$

$$t_\sigma t'_\sigma = 1 - r_\sigma^2, \tag{1.65i}$$

$$t_\pi t'_\pi = 1 - r_\pi^2. \tag{1.65j}$$

1.7.1.1 Brewster Angle

Fresnel relations reveal an interesting consequence of the boundary conditions. The reflection coefficient for σ-polarized light does not become zero for any angle of incidence, but the reflection coefficients for π- and n-polarizations vanish for angle of incidence θ_B, satisfying the condition

$$n_1 \cos\phi = n_2 \cos\theta_B. \tag{1.66a}$$

This result when combined with Snell's law gives

$$\phi = \frac{\pi}{2} - \theta_B. \tag{1.66b}$$

Accordingly, the reflection coefficient of light polarized in the plane of incidence becomes zero when the angle between the directions of propagation of the reflected and transmitted light waves becomes 90°. The angle of incidence θ_B satisfying this condition is known as Brewster angle. The π- and n-polarized waves at this angle of incidence do not undergo any reflection and are therefore fully transmitted. The σ-polarized light, on the other hand, is partially transmitted and partially reflected at all angles of incidence including the Brewster angle. Equations (1.66) give for the Brewster angle, the condition

$$\tan\theta_B = \frac{n_2}{n_1}. \tag{1.67}$$

If unpolarized light is incident at this angle, the reflected light appears in pure σ-polarization. However, for $n_2/n_1 = 1.5$, as for the air–glass interface, $\theta_B = 56.3°$, and only 15% of the incident energy appears in the reflected light. Notwithstanding this rather low polarizing efficiency, the Brewster angle is also known as the polarizing angle. Lasers make a very effective use of incidence at Brewster angle for controlling the state of polarization of laser light. This is shown in Fig. 1.8. Glass or quartz windows are fused to the plasma tube of a laser at both ends at the Brewster angle. At each of the four interfaces, σ-polarized

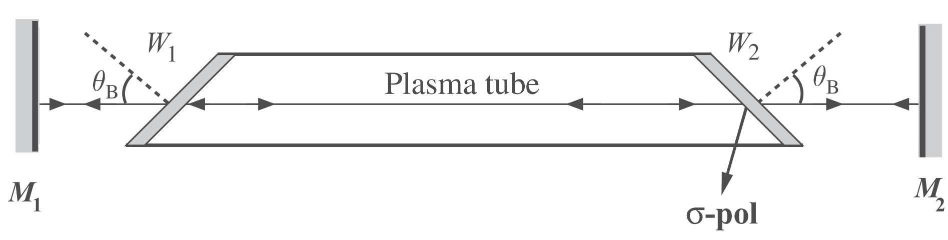


Fig. 1.8: Brewster windows (W_1, W_2) of the plasma tube of a laser.

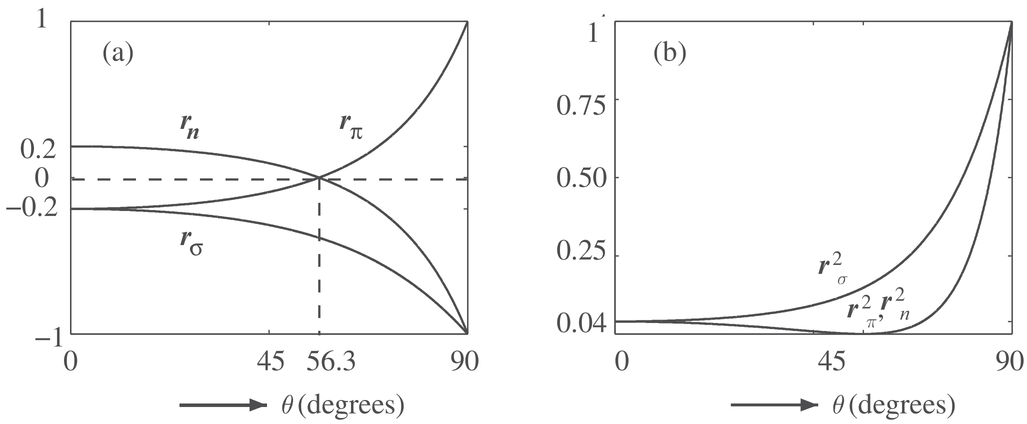


Fig. 1.9: Variations of reflection coefficients (a) and their squares (b) with angle of incidence for external reflections ($n_2/n_1 = 1.5$).

light suffers substantial (15% for glass windows) reflection losses whereas light polarized in the plane of incidence is transmitted without any reflection loss. The laser cavity (mirrors M_1, M_2 and the active medium filling the plasma tube) is unable to sustain oscillations for the σ-polarized light in the presence of these losses. Consequently, light coming out of a laser with Brewster windows is polarized in the plane of incidence. The σ-polarized light with electric field perpendicular to the plane of incidence is eliminated in the process.

Variations of the reflection coefficients and their squares with the angle of incidence are shown in Fig. 1.9 for the three states of polarization. The reflection coefficient is rather small at normal incidence (0.2 for $n_2/n_1 = 1.5$), but approaches unit value at grazing incidence ($\theta \longrightarrow 90°$). The three polarization states behave differently. The σ-polarized light suffers 180° phase change on reflection at all angles of incidence. The π-polarized light, however, undergoes phase reversal only up to the Brewster angle, and no phase change for incidence beyond this angle. The n-polarized light has the behavior just opposite to that of the π-polarized light (Fig. 1.9a).

The reflection coefficients and their squares vanish at the Brewster angle for π- and n-polarizations. The reflected light is richer in σ polarization, except for incidence at normal and grazing angles.

1.7.2 Reflectance and Transmittance

It was mentioned that the Fresnel relations in their present form (Eqs 1.62) may differ somewhat from Fresnel relations given elsewhere. The difference lies in the fact that we have decomposed the field vectors into three components along the $\hat{\pi}$-, $\hat{\sigma}$-, and $\hat{n}$-directions. In most texts, the in-plane ($\hat{\pi}$- and $\hat{n}$-) components are not separated. Instead, one deals with only two field components – the perpendicular or the σ-component and the parallel component which is the vector sum of the π- and n-components. In this context, we would like the readers to appreciate that the reflection and transmission amplitude coefficients may not always be useful quantities since the measurable quantities are the intensities and not the fields. The reflectance (or the reflectivity) R and transmittance (or the transmitivity) T, which refer to the division of the incident irradiance into the reflected and transmitted irradiances, are of fundamental significance. In the absence of absorption and scattering losses at the interface between two media, the relation

$$R + T = 1 \tag{1.68}$$

must hold for reasons of energy conservation. The incident, reflected and transmitted energies crossing per unit time per unit area of the interface are

$$\begin{aligned} I_{\mathrm{in}} &= \vec{S}_{\mathrm{i}} \,.\, \hat{n} = S_{\mathrm{i}} \cos\theta, \\ I_{\mathrm{re}} &= \vec{S}_{\mathrm{r}} \,.\, \hat{n} = S_{\mathrm{r}} \cos\theta, \\ I_{\mathrm{tr}} &= \vec{S}_{\mathrm{t}} \,.\, \hat{n} = S_{\mathrm{t}} \cos\phi, \end{aligned}$$

respectively. So that

$$R = \frac{I_{\mathrm{re}}}{I_{\mathrm{in}}} = \frac{S_{\mathrm{r}}\cos\theta}{S_{\mathrm{i}}\cos\theta} = \left(\frac{E_{\mathrm{r}}}{E_{\mathrm{i}}}\right)^2 = r^2, \tag{1.69a}$$

$$T = \frac{I_{\mathrm{tr}}}{I_{\mathrm{in}}} = \frac{S_{\mathrm{t}}\cos\phi}{S_{\mathrm{i}}\cos\theta} = \frac{n_2}{n_1}\frac{\cos\phi}{\cos\theta}\left(\frac{E_{\mathrm{t}}}{E_{\mathrm{i}}}\right)^2 = \frac{n_2}{n_1}\frac{\cos\phi}{\cos\theta}t^2, \tag{1.69b}$$

where S_{i}, S_{r}, and S_{t} are the magnitudes of the incident, reflected and transmitted Poynting vectors at the interface, respectively. For perpendicular (σ-) polarization,

$$R_\sigma = r_\sigma^2 = \left(\frac{n_1\cos\theta - n_2\cos\phi}{n_1\cos\theta + n_2\cos\phi}\right)^2, \tag{1.70a}$$

$$T_\sigma = \frac{n_2}{n_1}\frac{\cos\phi}{\cos\theta}t_\sigma^2 = \frac{n_2}{n_1}\frac{\cos\phi}{\cos\theta}\left(\frac{2n_1\cos\theta}{n_1\cos\theta + n_2\cos\phi}\right)^2. \tag{1.70b}$$

It can be seen that the condition

$$R_\sigma + T_\sigma = 1$$

holds. For the in-plane or the so-called parallel polarization, we need to combine the n- and π-components since they do not represent independent waves. Therefore,

$$\begin{aligned} R_{\mathrm{p}} &= r^2 \text{(parallel polarization)} \\ &= \frac{E_{\mathrm{r}n}^2 + E_{\mathrm{r}\pi}^2}{E_{\mathrm{i}n}^2 + E_{\mathrm{i}\pi}^2} \\ &= \frac{r_n^2 E_{\mathrm{i}n}^2 + r_\pi^2 E_{\mathrm{i}\pi}^2}{E_{\mathrm{i}n}^2 + E_{\mathrm{i}\pi}^2} \\ &= \left(\frac{n_1 \cos\phi - n_2 \cos\theta}{n_1 \cos\phi + n_2 \cos\theta}\right)^2 \end{aligned} \tag{1.71a}$$

and

$$\begin{aligned} T_{\mathrm{p}} &= \frac{n_2}{n_1}\left(\frac{E_{\mathrm{t}n}^2 + E_{\mathrm{t}\pi}^2}{E_{\mathrm{i}n}^2 + E_{\mathrm{i}\pi}^2}\right)\left(\frac{\cos\phi}{\cos\theta}\right) \\ &= \frac{4 n_1 n_2 \cos\phi \cos\theta}{(n_1 \cos\phi + n_2 \cos\theta)^2}. \end{aligned} \tag{1.71b}$$

Once again, it can be seen that $R_{\mathrm{p}} + T_{\mathrm{p}} = 1$. Figure 1.10 shows the variations in the reflectance and transmittance with the angle of incidence for external reflections ($n_2/n_1 = 1.5$).

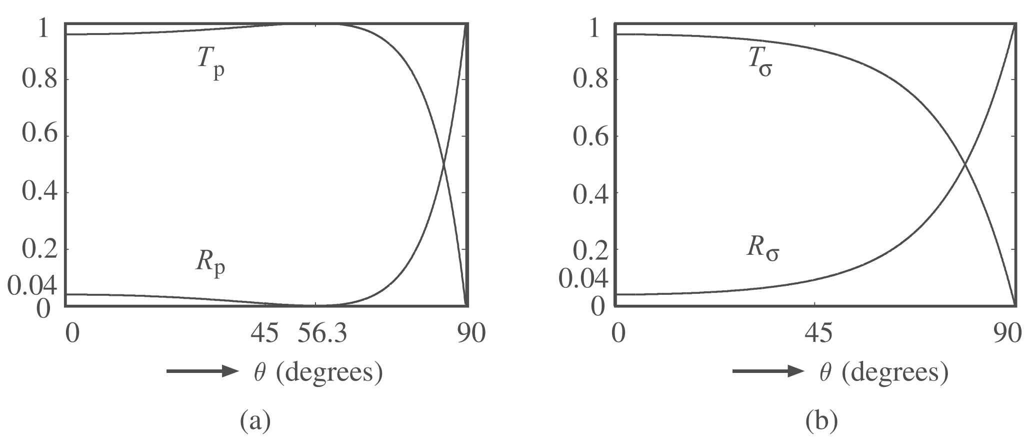


Fig. 1.10: Reflectance and transmittance changes with angle of incidence for external reflections ($n_2/n_1 = 1.5$); (a) parallel or in-plane polarization, (b) perpendicular polarization.

1.7.3 Internal Reflections

When the refractive index n_2 of the second medium is lower than the refractive index n_1 of the first medium, the right-hand side of Eq. (1.63) cannot remain real for all angles of incidence. Beyond a certain angle of incidence, called the critical angle θ_c defined by

$$n_2 = n_1 \sin\theta_c, \tag{1.72}$$

the right-hand side becomes purely imaginary. For incident angles smaller than the critical angle, the attenuation vector $\vec{a}_t$ vanishes as the right-hand side is real, and the transmitted wave in the second medium is homogeneous with the magnitude of the wave vector $k_t = n_2\omega/c$, just as for the external reflections. Except for the fact that the angle of refraction exceeds the angle of incidence, there is no qualitative difference in external and internal reflections as long as the angle of incidence remains smaller than the critical angle. In fact, the π- and n-polarizations go through zero reflectivity at the corresponding Brewster angle in this case as well. Brewster angle is always smaller than the critical angle (for $n_1/n_2 = 1.5$, $\theta_B = 33.7°$ and $\theta_c = 41.8°$). However, the situation changes non-trivially as the critical angle is approached. At the critical angle, the right-hand side of Eq. (1.63) vanishes, forcing $k_t \cos\phi$ and a_t to take zero values. This happens when the angle of refraction ϕ becomes 90° and wave propagation in the second medium takes place along the interface only (Fig. 1.11). Equations (1.65) give reflection coefficient of unit magnitude at this angle of incidence, irrespective of the state of polarization. Light is therefore totally reflected back into the first medium; hence the use of the term total internal reflection to describe wave propagation from an optically dense to an optically rare medium for angles of incidence at and beyond the critical angle. It may

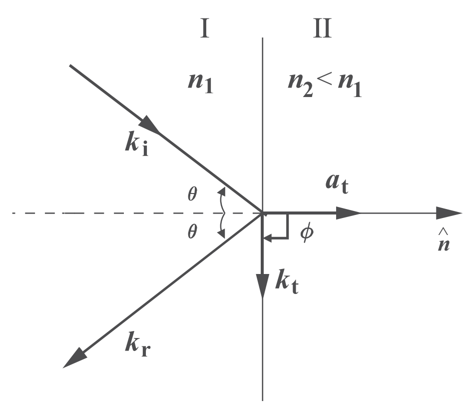


Fig. 1.11: Geometry for internal reflections. Wave in second medium is inhomogeneous for angles of incidence exceeding the critical angle.

appear confusing that the wave is totally reflected back into the first medium despite wave propagation taking place along the interface ($\phi = 90°$). We shall return to this point shortly. The wave propagating along the interface is called the evanescent (tending to vanish) wave.

As the angle of incidence exceeds the critical angle, the right-hand side of Eq. (1.63) becomes purely imaginary and the transmitted wave continues to propagate along the interface with propagation vector $\vec{k}_t$ of magnitude (Eq. 1.53b)

$$k_t = n_1 \frac{\omega}{c} \sin\theta, \tag{1.73}$$

but now with an attenuation vector $\vec{a}_t$ of magnitude

$$a_t = \frac{\omega}{c}(n_1^2 \sin^2\theta - n_2^2)^{1/2} \tag{1.74}$$

directed normal (Eq. 1.51e) to the plane of the boundary (Fig. 1.11). Equation (1.49c) for the transmitted wave now takes the form

$$\vec{E}_{tr} = \vec{E}_t\, e^{i[(\vec{k}_t + i\vec{a}_t)\cdot\vec{r} - \omega t]}. \tag{1.75}$$

Substituting k_t and a_t from Eqs (1.73) and (1.74) gives

$$\vec{E}_{tr} = \vec{E}_t\, e^{-\frac{\omega}{c}(n_1^2\sin^2\theta - n_2^2)^{1/2} z}\, e^{i(\frac{n_1\omega}{c} x \sin\theta - \omega t)}. \tag{1.76}$$

The transmitted wave (evanescent wave) propagates in the x direction. The amplitude of the wave in the second medium decreases exponentially with z, falling to $1/e$ of its value at the interface at a distance

$$\delta = \frac{1}{a_t} = \frac{\lambda_v}{2\pi(n_1^2\sin^2\theta - n_2^2)^{1/2}} \tag{1.77}$$

away from the interface. The beam attenuation increases with increasing angle of incidence beyond the critical angle. For the glass–air interface, $\delta = 2.3 \times 10^{-5}$ cm for $\theta = 45°$ and $\lambda_v = 500$ nm. The penetration depth δ in the second medium is only a fraction of the wavelength of light. The surfaces of constant phase (normal to $\vec{k}_t$) are normal to the plane of the interface and the surfaces of constant amplitude (normal to $\vec{a}_t$) are parallel to the plane of the interface.The evanescent wave in the second medium is therefore an inhomogeneous wave with the phase velocity ($\omega/k_t = c/(n_1 \sin\theta)$) exceeding the velocity of light (c/n_1) in the medium. Total internal reflection makes it possible for light to propagate in optical fibers and optical wave guides.

The reflection and transmission coefficients for internal reflections for $\theta < \theta_c$ are still given by Eqs (1.65), just as for the external reflections. But now n_2 being smaller than n_1, the signs of the reflection coefficients are opposite to those for the external reflections. For $\theta > \theta_c$, Eqs (1.62) give the following expressions for the reflection coefficients:

$$r_\sigma = \frac{n_1 \cos\theta - \mathrm{i}(n_1^2 \sin^2\theta - n_2^2)^{1/2}}{n_1 \cos\theta + \mathrm{i}(n_1^2 \sin^2\theta - n_2^2)^{1/2}} = \mathrm{e}^{-\mathrm{i}2\phi_0}, \tag{1.78a}$$

$$r_\pi = \frac{-n_2^2 \cos\theta + \mathrm{i}n_1(n_1^2 \sin^2\theta - n_2^2)^{1/2}}{n_2^2 \cos\theta + \mathrm{i}n_1(n_1^2 \sin^2\theta - n_2^2)^{1/2}} = \mathrm{e}^{-\mathrm{i}(2\psi_0+\pi)}, \tag{1.78b}$$

where

$$\tan\phi_0 = \frac{(n_1^2 \sin^2\theta - n_2^2)^{1/2}}{n_1 \cos\theta}, \tag{1.79a}$$

$$\tan\psi_0 = \left(\frac{n_1}{n_2}\right)^2 \frac{(n_1^2 \sin^2\theta - n_2^2)^{1/2}}{n_1 \cos\theta}. \tag{1.79b}$$

The reflection coefficients are now complex with unit magnitude for any state of polarization for all angles exceeding the critical angle. The reflection is therefore total. For internal reflections, the variations of the reflection coefficients and reflectances with the angle of incidence are shown in Fig. 1.12. The phase changes for the reflected fields are different for the π- and σ-polarizations. Accordingly, linearly polarized light, polarized along directions other than $\hat{\pi}$- and $\hat{\sigma}$-directions, becomes elliptically polarized after an internal reflection. The phase for σ-polarization changes from $2\phi_0 = 0$ at $\theta = \theta_c$ to $2\phi_0 = \pi$ at $\theta = 90°$. The π-polarization, on the other hand, undergoes a 180° phase change (change

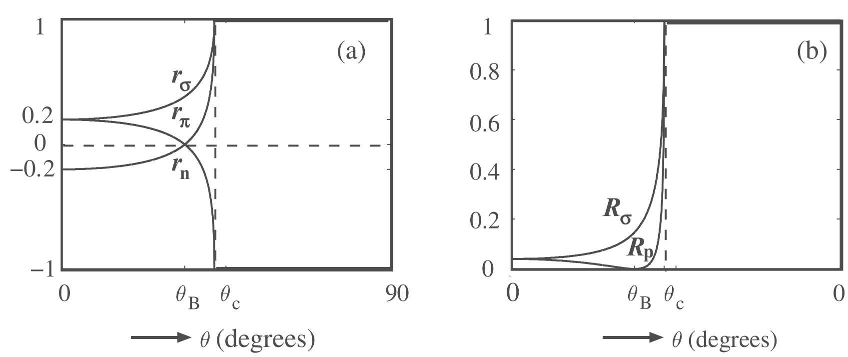


Fig. 1.12: Variation of reflection coefficients (a) and reflectances (b) with angle of incidence for internal reflection ($n_1/n_2 = 1.5$); $\theta_B = 33.7°$, $\theta_c = 41.8°$.

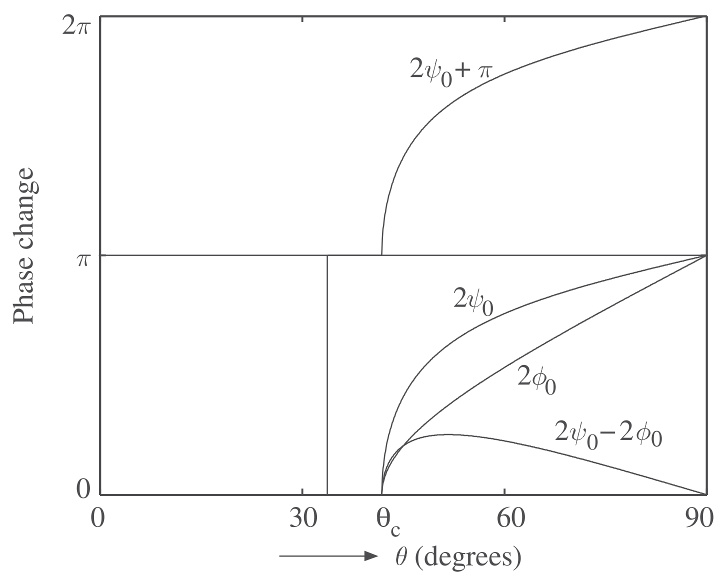


Fig. 1.13: Phase changes during internal reflections with angle of incidence $(n_1/n_2 = 1.5)$; ψ_0 is for π-polarization and ϕ_0 is for σ-polarization.

of sign) at the Brewster angle. Additional phase changes take place beyond the critical angle. The net phase of π-polarized wave varies from $2\psi_0 + \pi = \pi$ at $\theta = \theta_c$ to $2\psi_0 + \pi = 2\pi$ at $\theta = 90°$. These phase changes are shown in Fig. 1.13. The same figure also shows the variations of $2\psi_0$ and $2\psi_0 - 2\phi_0$.

The phase difference $2(\psi_0 - \phi_0)$ between π- and σ-polarizations can be obtained from

$$\tan(\psi_0 - \phi_0) = \frac{\cos\theta}{\sin^2\theta}\left(\sin^2\theta - \frac{n_2^2}{n_1^2}\right)^{1/2}. \tag{1.80}$$

For $n_1/n_2 = 1.5$, the maximum value of $(2\psi_0 - 2\phi_0)$ of 45.2° occurs at $\theta = 54°$.

1.7.3.1 Fresnel Rhomb

This device, first conceived by Fresnel, is used to change the state of polarization of light from linear to circular by introducing a phase difference of 90° between the π- and σ-polarized light waves through two successive internal reflections in a rhomb, cut with an apex angle which allows 45° phase change in each internal reflection (Fig. 1.14). The incident beam, linearly polarized at 45° with the face edge, enters the rhomb normally. The beam suffers two internal reflections inside the rhomb and leaves through the opposite face of the rhomb normally, but now circularly polarized. Unlike a quarter-wave plate

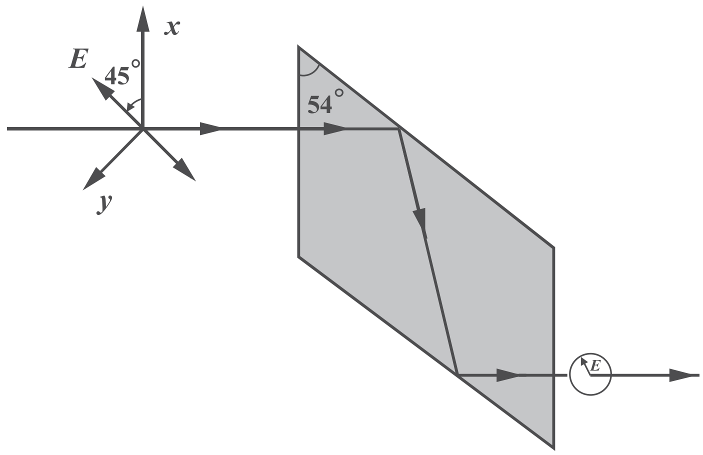


Fig. 1.14: Fresnel rhomb to convert linearly polarized light into circularly polarized light.

(see Section 3.3.2), Fresnel rhomb is much less sensitive to changes in the wavelength of light.

1.7.4 Frustrated Total Internal Reflection

We have seen that despite the existence of the evanescent wave along the interface, light is fully reflected back into the first medium. Consequently, no energy can flow into the second medium. This, as a matter of fact, is a correct statement and can be proved by showing that the time averaged value of the z-component of the Poynting vector in the semi-infinite second medium is actually zero. This, however, does not fully clarify the situation. There is a need to further explore what actually happens in the neighborhood of the interface. It has already been mentioned that light does penetrate into the second medium, but the depth of penetration is rather small. This can be verified. Consider a thin slab of lower refractive index n_2 sandwiched between thicker slabs of a medium of higher refractive index n_1 as shown in Fig. 1.15.

Let the thickness d of the sandwiched slab be comparable to the penetration depth of the wave. For incidence at the first interface at an angle greater than the critical angle, the transmitted wave can be detected beyond the second interface. The amplitude of the transmitted wave depends on the actual thickness of the sandwiched slab; thinner the sandwiched slab, larger the amplitude of the transmitted wave. However, to avoid multiple reflections in the sandwiched medium, its thickness should be somewhat larger than the penetration depth δ. It is therefore clear that notwithstanding what has been said earlier, light is partially transmitted in an internal reflection. However, if the thickness of the sandwiched slab is made sufficiently large, the transmitted wave after travelling

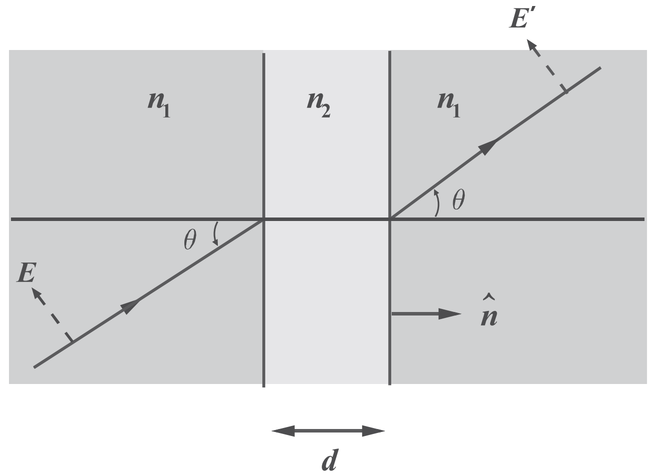


Fig. 1.15: Geometry to frustrate total internal reflection ($n_2 < n_1$).

a short distance in this medium apparently bends and re-enters the first medium, somewhat shifted from the position of entry into the second medium (Goos-Hanchan shift [1.4]). Thus, no net energy flows into the second medium making internal reflection total, indeed. But the time averaged component of the Poynting vector along the interface is non-zero (the evanescent wave). It is possible to frustrate the total internal reflection (make it less than total) by reducing the thickness of the middle slab. Arrangements of the type shown in Fig. 1.15 can control the amount of energy being coupled from one medium to the other. For σ-polarized light of amplitude E_σ entering the first interface, amplitude of the wave leaving the second interface (see Eq. 1.65i), is

$$\begin{aligned} E'_\sigma &= t_\sigma t'_\sigma \, \mathrm{e}^{-d/\delta} E_\sigma \\ &= (1 - r_\sigma^2)\, \mathrm{e}^{-d/\delta} E_\sigma \\ &= (1 - \mathrm{e}^{-\mathrm{i}4\phi_0})\, \mathrm{e}^{-d/\delta} E_\sigma, \end{aligned}$$

where ϕ_0 is as defined in Eq. (1.78a), δ the penetration depth (Eq. 1.77) and d the thickness of the sandwiched slab. It must be mentioned that bringing in the second interface as in Fig. 1.15 changes the original problem altogether. The boundary conditions at the first interface get modified due to the presence of the second interface.

We end this discussion by recalling that the external and internal reflections have been investigated here under the assumption of perfect transparency of the media on the two sides of the interface. Real optical materials are not perfectly transparent. For sufficiently high transparency ($\kappa \to 0$), the results obtained in this chapter may be used as such or with slight modification. For example, complete absence of π-polarized light on reflection at Brewster angle may not happen in real optical materials. Instead, the reflection coefficient for π-polarized light goes through a sharp minimum at this angle. Similar modifications may be expected elsewhere.

1.7.5 Reflection from a Metallic Surface

The formalism developed in the preceding sections can describe reflection from a metallic surface. However, the wave equation applicable to metals is quite different from the one developed in this chapter because the free charge and free currents appearing in Maxwell's equations do not vanish for metals. Nevertheless, it is possible to gain some insight of wave propagation in metals from Fresnel relations if allowance is made for absorption to take place in the second medium [1.4, 1.5]. Metals are generally opaque to visible light unless thin metallic films no more than a few nanometers (10^{-7} cm) in thickness are employed. Special care needs to be exercised for the preparation of thin metallic films if they are to faithfully represent the behavior of bulk metals. Thin metallic films are partially transparent in some regions of the visible spectrum. For example, gold and copper with yellow luster are somewhat transparent to blue-green light if used in the form of thin films. Table 1.2 gives real and imaginary parts of the index of refraction of some metals in the visible region.

For good conductors, the imaginary part of the refractive index is much larger than the real part, and an approximate expression

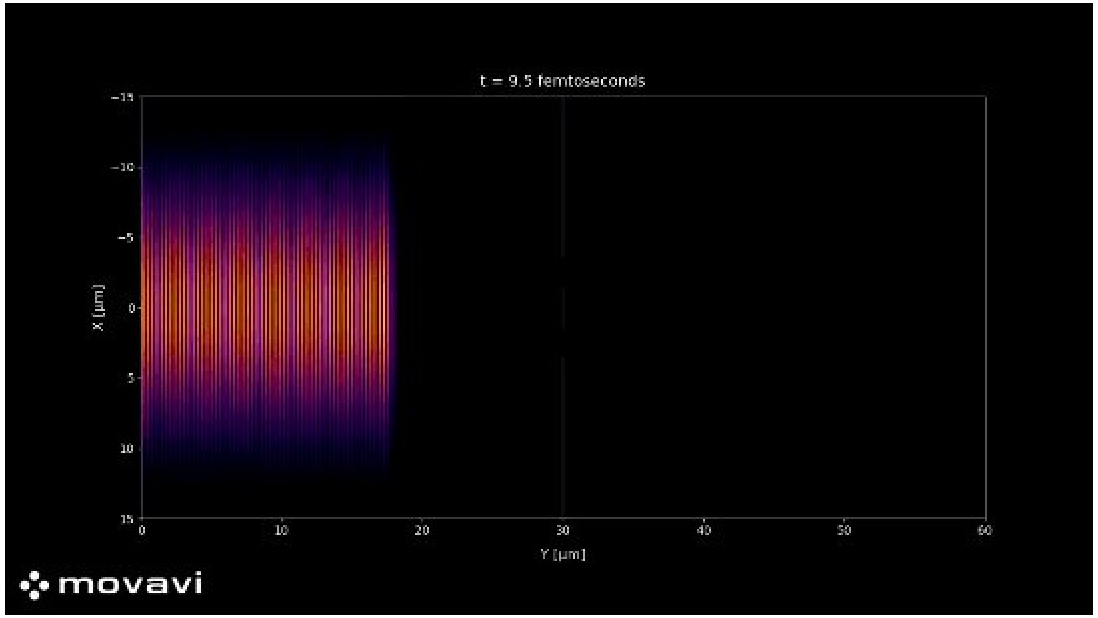
$$\sin\phi = \frac{\sin\theta}{\kappa} \tag{1.81}$$

holds for the angle of refraction ϕ, where θ is the angle of incidence. For incidence at 60°, the angle of refraction for aluminum is merely 7°. Thus for good conductors, the transmitted wave propagates essentially along the normal to the plane of the interface. The propagation vector $\vec{k}_t$ and attenuation vector $\vec{a}_t$ are nearly coincident. Therefore, the wave in a good conductor is very nearly

Table 1.2. Complex refractive index $\tilde{n} = n + i\kappa$ of some metals.

Metal	λ (nm)	n	κ
Al	650	1.30	7.11
Pd	550	1.8	4.0
Cu	548	0.76	2.46
Ag	584	0.055	3.32
Na	546	0.05	2.20
Au	546	0.4	2.3

The Spatial Coherence



→ Purely coherent wave

Interferometric visibity of the fringes for a highly coherent wave

$$\gamma = \frac{I_{max} - I_{min}}{I_{max} + I_{min}}$$

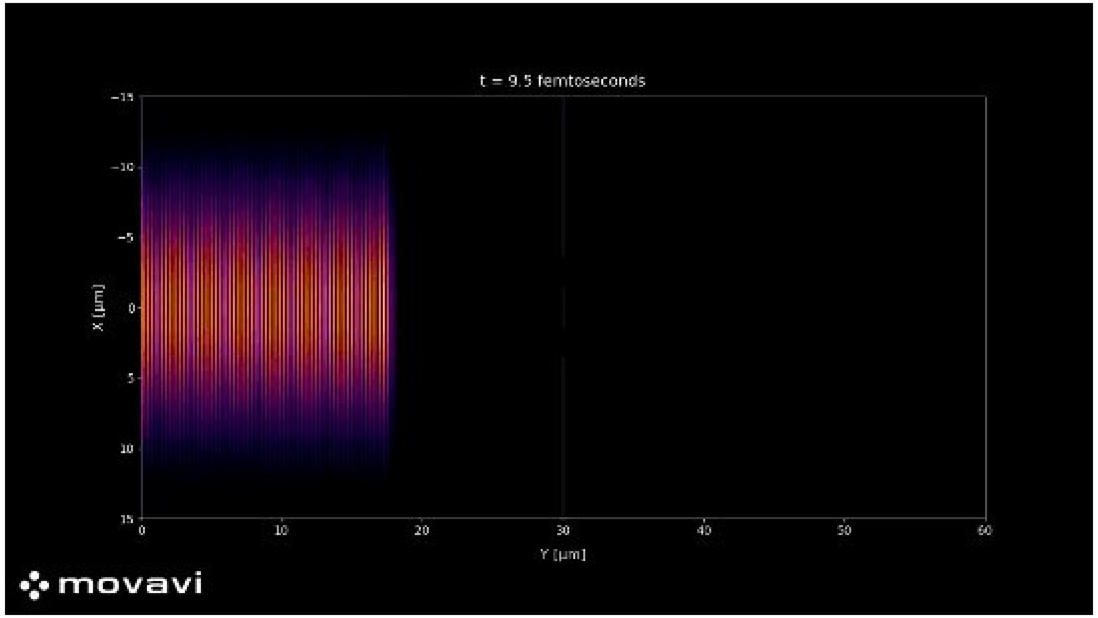
≈ 1

→ simulation showing the spatial interference and the visibility of the fringes in the young's double slit expt,

→ each wavefronts of the interfered light have definite (constant) phase, hence the spatial stability of the interfered fringes over the distance.

The light intensity of the interfered fringes are averaged over a few picosecond time

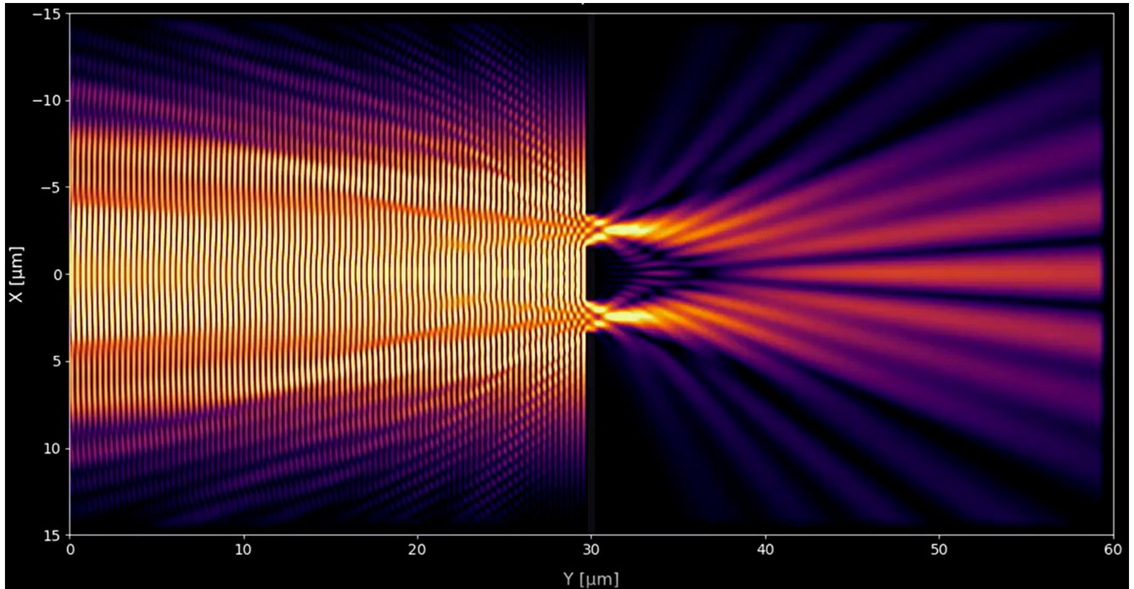
The Spatial Coherence

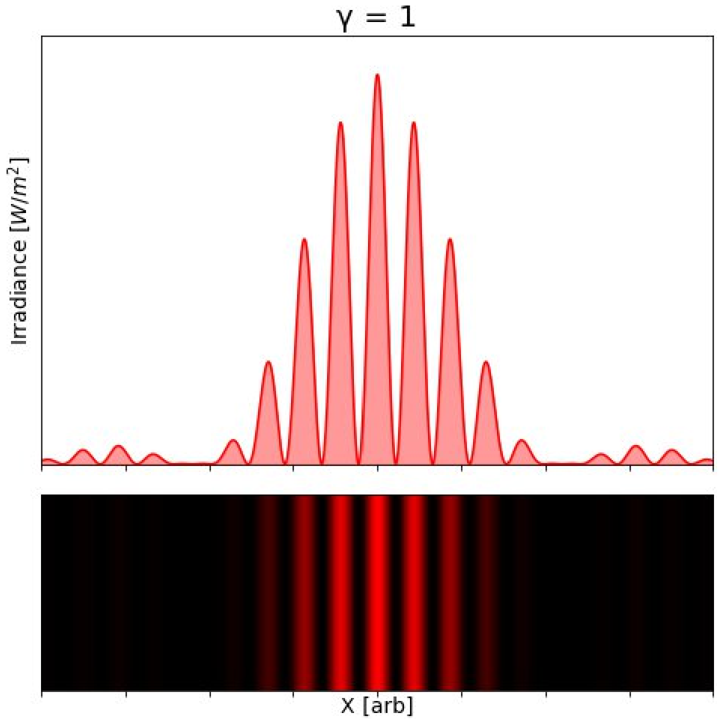


For a highly (purely
coherent wave

$$\gamma = \frac{I_{max} - I_{min}}{I_{max} + I_{min}} = 1$$

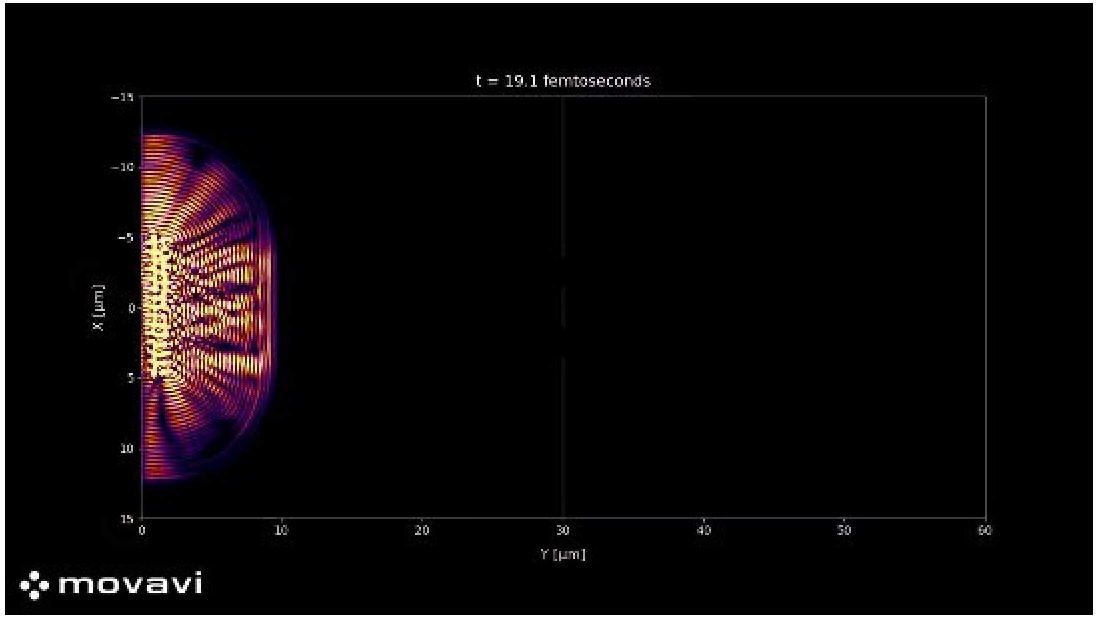
$\Rightarrow$ 100% visibility of the
interfered fringes.

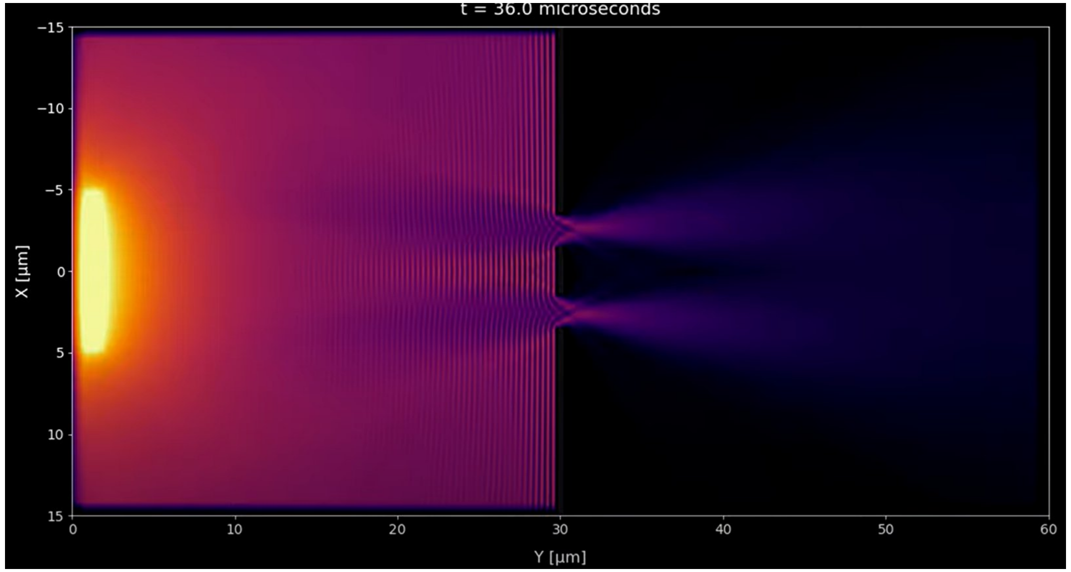




https://rafael-fuente.github.io/

The Spatial Coherence





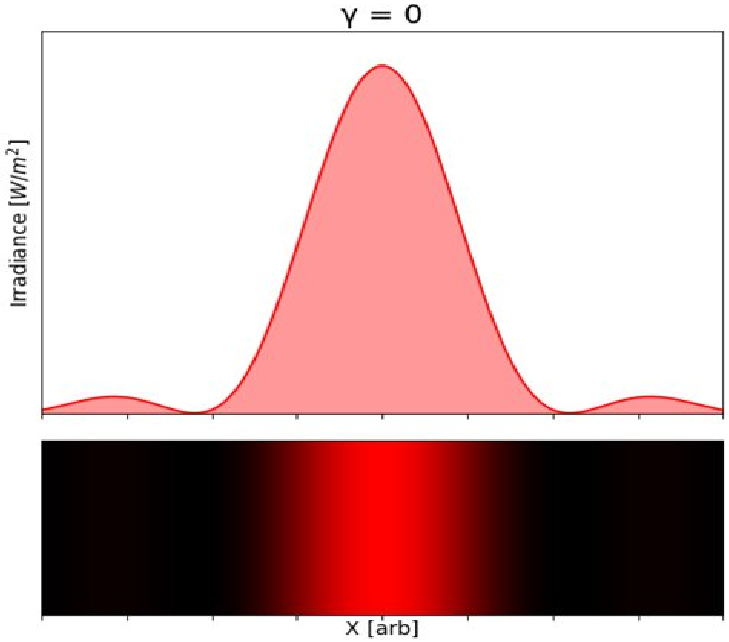
→ Incoherent light wave

→ Extended source (No phase correlations)

→ Fringe intensity averaged over milliseconds

→ Interferometric visibility of the fringes $\gamma \approx 0$

→ No interference!



https://rafael-fuente.github.io/

QT209: Classical Optics: Sample numerical questions for practice

1. Two waves of wavelength 1.53 um and 1.56 um are travelling in a medium of refractive index 1.50 and 1.49 respectively. Calculate the group velocity of the waves in the respective medium. Consider velocity of wave in vacuum 3 x 10^8 m/sec

2. Calculate the wave number for the light of wavelength 632nm. Calculate the phase and group velocity of the wave when its angular frequency is twice the wavenumber.

3. For calcite the values of n_0 and n_e for the wave of λ_0 = 500nm are 1.6 and 1.5 respectively, and corresponding to the wave of λ_0 = 1000nm, the n_0 = 1.58 and ne = 1.48. The calcite works as a quarter wave plate for the wave λ_0 = 500nm. If a left circularly polarized beam of λ_0 = 1000nm falls on this plate, then calculate the phase and the state of polarization of the emergent beam.

4. A Michelson interferometer is illuminated by a light source of spectral width $\Delta v=6\times10^{10}$ Hz. Calculate the maximum path difference for which interference fringes are still visible.

5. Explain how spatial coherence depends on the size of the source and the aperture used in the experiment.

6. Calculate the numerial aperture of a step index fiber having refractive index n1=1.48 and n2=1.46. What is the maximum launching angle for this fiber if the outer medium is air?

7. A certain optical fiber has an attenuation of 0.6dB/km at 1300nm and 0.3dB/km at 1550 nm. Suppose these two optical signals are launched simultaneously into the fiber: an optical power of 150uW at 1300 nm and optical power of 100uW at 1550 nm. What are the power level in uW of these two signals at a) 5 km b)25km.

Practice questions:

1. Unpolarized light passes through a linear polarizer with its transmission axis along the y-axis. The output intensity of the light is I_0. This light then passes through a second polarizer with its transmission axis at an angle of 60^0 to the y-axis. Find the final intensity of the light after passing through both polarizers.

2. Consider an electromagnetic wave incident on a boundary between air and a dielectric material with a refractive index n= 2. If the incident angle θ_i=45^0 and the wavelength in air λ_0=600 nm, calculate the penetration depth of the inhomogeneous wave inside the air medium.

3. Determine the core radius required for a fiber with wavelength λ=1.31 μm to ensure single-mode operation, given that core refractive index n_1=1.52 and Δn=0.02

4. If an optical pulse of width 0.5 ps experiences a dispersion of 10 ps/(nm km) over 200 km, what is the new width of the pulse?

5. A laser with a wavelength of 600 nm and spectral width of 0.2 nm has coherence time of ____________________________.